E_k-algebras and homological stability

Oscar Randal-Williams

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Premise

Want to study the homology of things like $GL_n(\mathbb{k})$, in particular its behaviour with respect to varying n.

Have stabilisation maps

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : GL_{n-1}(\mathbb{k}) \longrightarrow GL_n(\mathbb{k})$$

and *homological stability* hopes these are homology isomorphisms in a range of degrees going to ∞ with n.

Equivalently, it hopes that

$$H_d(GL_n(\mathbb{k}), GL_{n-1}(\mathbb{k})) = 0 \text{ for all } d \leq f(n)$$

for some divergent function f.

Reformulation

The space

$$\mathbf{R}^+ = \bigsqcup_{n \geq 0} BGL_n(\mathbb{k})$$

is a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces. Write $H_{n,d}(\mathbf{R}^+) := H_d(BGL_n(\mathbb{k}))$ for homology in this category.

For the basepoint $\sigma \in H_0(BGL_1(\mathbb{k}))$ the stabilisation map can be described in terms of the E_∞-multiplication as

$$- \cdot \sigma : H_d(BGL_{n-1}(\mathbb{k})) \longrightarrow H_d(BGL_n(\mathbb{k})).$$

Writing $\mathbf{R}^+/\sigma$ for the cofibre in graded spaces of $- \cdot \sigma : \mathbf{R}^+[1] \to \mathbf{R}^+$,

$$H_d(GL_n(\mathbb{k}), GL_{n-1}(\mathbb{k})) = H_{n,d}(\mathbf{R}^+/\sigma).$$

Goal: Exploit the E_∞-structure on $\mathbf{R}^+$ to analyse homological stability.

Everything is based on joint work with S. Galatius and A. Kupers.

Homotopy theory of E_k-algebras

Graded objects

Let C denote sSet, sSet_*, Sp, or (because we are eventually interested in taking $\Bbbk$-homology) $\mathrm{sMod}_{\Bbbk}$.

Write $\otimes$ for the cartesian, smash, or tensor product.

We will consider $\mathbb{N}$-graded objects in C, meaning $\mathrm{C}^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, \mathrm{C})$. This is given the Day convolution monoidal structure:

$$(X \otimes Y)(n) = \bigsqcup_{a+b=n} X(a) \otimes Y(b).$$

Define bigraded homology groups as $H_{n,d}(X; \Bbbk) := H_d(X(n); \Bbbk)$.

Define graded spheres in $\mathrm{sSet}^{\mathbb{N}}$ as

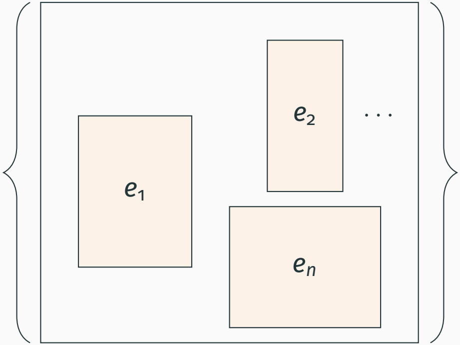
$$S^{n,d}(m) = \begin{cases} S^d & \text{if } n = m \\ \varnothing & \text{else} \end{cases}$$

and similarly discs $D^{n,d}$. Analogously in the other categories.

E_k-algebras

Let $\mathcal{C}_k$ denote the non-unital ($\mathcal{C}_k(0) = \varnothing$) little k-cubes operad.

$\mathcal{C}_2(n) =$



The categories $\mathsf{C}^{\mathbb{N}}$ are all tensored over Top: can make sense of the monad

$$E_k(X) := \bigsqcup_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n}$$

and so of E_k-algebras $\mathbf{X}$ in $\mathsf{C}^{\mathbb{N}}$. Call the category of these $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$.

E_k-cells

Each $\mathsf{C}^{\mathbb{N}}$ may be given the levelwise model structure, and $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$ then has the projective model structure, making

$$X \mapsto \mathbf{E_k}(X) : \mathsf{C}^{\mathbb{N}} \rightleftarrows \mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}}) : X \mapsfrom \mathbf{X}$$

a Quillen adjunction.

Given an E_k-algebra $\mathbf{X}$ and a map $f : S^{n,d-1} \to X$ can define the cell attachment $\mathbf{X} \cup_f^{E_k} D^{n,d}$ as the pushout in $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$ of

$$\mathbf{E_k}(D^{n,d}) \longleftarrow \mathbf{E_k}(S^{n,d-1}) \xrightarrow{f^{ad}} \mathbf{X}.$$

Cellular E_k-algebras are those formed by iterated cell attachments.

(Every object is equivalent to a cellular one, as usual.)

Filtrations

Let $\mathsf{D} := \mathsf{C}^{\mathbb{N}}$ and $\mathbb{Z}_{\leq}$ be the poset of integers. A *filtered object* in D is a functor $\mathbb{Z}_{\leq} \to \mathsf{D}$, and $\mathsf{D}^{\mathbb{Z}_{\leq}}$ is the category of such.

The *underlying object* of a filtered X is $\operatorname{colim}_{\mathbb{Z}_{\leq}} X \in \mathsf{D}$.

The *filtration quotients* of a filtered X are the cofibres, i.e. the pointed objects given by the pushouts

$$* \longleftarrow X(n-1) \longrightarrow X(n).$$

Taking associated graded gives a strong monoidal functor

$$\mathrm{gr} : \mathsf{D}^{\mathbb{Z}_{\leq}} \longrightarrow \mathsf{D}_*^{\mathbb{Z}}.$$

If X is cofibrant have a spectral sequence

$$E^1_{n,p,q} = \widetilde{H}_{n,p+q}(\mathrm{gr}(X)(q)) \Rightarrow H_{n,p+q}(\operatorname{colim} X).$$

A *filtered E_k-algebra in* D is an E_k-algebra in $\mathsf{D}^{\mathbb{Z}_{\leq}}$.

A *CW-E_k-algebra* is (roughly) a cellular object in filtered E_k-algebras, where the attaching maps of the d-cells have filtration $\leq d-1$.

Indecomposables

For $\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}})$ define the E_k-*indecomposables* of $\mathbf{X}$ by

$$E_k(X) = \bigsqcup_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n} \underset{c}{\overset{\mu_X}{\rightrightarrows}} X \longrightarrow Q^{E_k}(\mathbf{X})$$

where c collapses all factors with $n > 1$ to the basepoint, and applies the augmentation $\varepsilon : \mathcal{C}_k(1)_+ \to S^0$.

Q^{E_k} is left adjoint to the inclusion $\mathsf{C}_*^{\mathbb{N}} \to \mathrm{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}})$ by imposing the trivial E_k-action.

Have $Q^{E_k}(\mathbf{E_k}(X)) = X$ (the coequaliser is split).

If $\mathbf{X}$ is a cellular E_k-algebra then it follows that $Q^{E_k}(\mathbf{X})$ is a cellular object with a (g, d)-cell for each E_k-(g, d)-cell of $\mathbf{X}$.

If $\mathbf{X}$ is not cofibrant we should instead evaluate the derived functor

$$Q^{E_k}_{\mathbb{L}}(\mathbf{X}) := Q^{E_k}(\text{cofibrant replacement of } \mathbf{X}).$$

AKA topological Quillen homology (for the operad $\mathcal{C}_k$).

E_k-homology and minimal cell structures

Define E_k-homology as $H^{E_k}_{n,d}(\mathbf{X}) := H_{n,d}(Q^{E_k}_{\mathbb{L}}(\mathbf{X}))$.

If $\mathbb{k}$ is a field, the discussion so far shows

$$\dim_{\mathbb{k}} H^{E_2}_{n,d}(\mathbf{X}; \mathbb{k}) \leq \begin{array}{c}\text{number of } E_2\text{-}(n,d)\text{-cells in any} \\ E_2\text{-cellular approximation of } \mathbf{X}.\end{array}$$

Just as in classical homotopy theory, homology can be used to detect *minimal* cell structures as long as we work in a stable context.

The following will suffice for now.

Theorem. Let $\mathbb{k}$ be a field and C be the category of simplicial $\mathbb{k}$-modules (or $H\mathbb{k}$-module spectra). Then $\mathbf{X} \in \mathrm{Alg}_{E_2}(\mathsf{C}^{\mathbb{N}})$ has a cellular approximation $c\mathbf{X} \xrightarrow{\sim} \mathbf{X}$ with $\dim_{\mathbb{k}} H^{E_2}_{g,d}(\mathbf{X})$-many E_2-(g,d)-cells.

Furthermore $c\mathbf{X}$ can be taken to be "CW", not just "cellular".

Computing E_k-homology

$Q_{\mathbb{L}}^{E_k}(\mathbf{X})$ may also be computed by a k-fold bar construction.

Instances of this have previously been given by Getzler–Jones, Basterra–Mandell, Fresse, Francis.

In particular, if $\mathbf{X}$ is an E_1-algebra it can be rectified to a nonunital associative algebra $\overline{\mathbf{X}}$ and unitalised to an associative algebra $\overline{\mathbf{X}}^+$. This unitalisation has an augmentation $\varepsilon : \overline{\mathbf{X}}^+ \to \mathbb{1}$. Then there is an equivalence

$$\mathbb{1} \vee \Sigma Q_{\mathbb{L}}^{E_1}(\mathbf{X}) \simeq B(\mathbb{1}; \overline{\mathbf{X}}^+; \mathbb{1})$$

with the two-sided bar construction.

(Something similar can be done for all E_k.)

From this perspective it is easy to see that vanishing lines for E_1-homology imply vanishing lines for E_2-homology, and so on.

The mapping class group E_2-algebra

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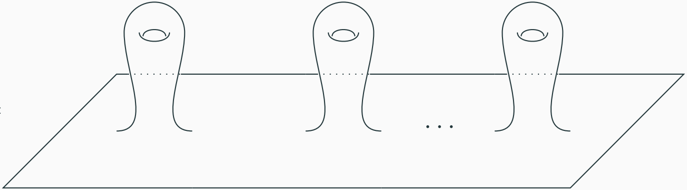
The surface

$\Sigma_{g,1} =$



has a mapping class group

$$\Gamma_{g,1} = \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,1})).$$

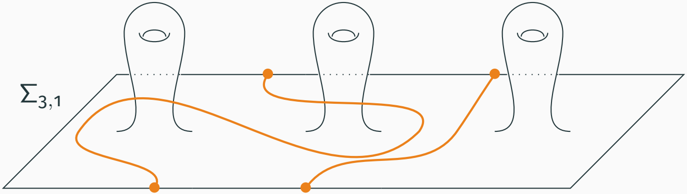
The collection $\bigsqcup_{g \geq 0} \Gamma_{g,1}$ has the structure of a braided monoidal groupoid, so taking nerves gives a unital E_2-algebra $\mathbf{R}^+$ in $\mathsf{sSet}^{\mathbb{N}}$ with

$$\mathbf{R}^+(g) \simeq B\Gamma_{g,1}.$$

Write $\mathbf{R}^+_{\Bbbk} \in \mathsf{Alg}_{E_k^+}(\mathsf{sMod}_{\Bbbk}^{\mathbb{N}})$ for its $\Bbbk$-linearisation.

A vanishing line for E_2-homology

The bar construction model for $Q_{\mathbb{L}}^{E_1}(\mathbf{R})$ leads us to study the simplicial complex whose p-simplices are $(p+1)$ arcs on the surface $\Sigma_{g,1}$, which cut it into $(p+2)$ components each of which have non-zero genus.



This is analogous to the Tits building of a vector space. We show that this simplicial complex is $(g-3)$-connected, and deduce

Theorem (Galatius–Kupers–R-W). $H_{g,d}^{E_2}(\mathbf{R}) = 0$ for $d < g - 1$.

Thus there is an E_2-cellular approximation $\mathbf{C} \xrightarrow{\sim} \mathbf{R}$ only having (g,d)-cells for $d \geq g - 1$.

Data

Many calculations of $H_d(\Gamma_{g,1})$ available for small g and d through the efforts of many mathematicians: Abhau, Benson, Bödigheimer, Boes, F. Cohen, Ehrenfried, Godin, Harer, Hermann, Korkmaz, Looijenga, Meyer, Morita, Mumford, Pitsch, Sakasai, Stipsicz, Tommasi, Wang, ...

d/g	0	1	2	3	4	5	6
2			$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1		$\mathbb{Z}$	$\mathbb{Z}/10$				
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

(Rows eventually constant = homological stability!)

However need more refined information than just abstract groups: E_2-structure, as encoded by multiplication $- \cdot -$, Browder bracket $[-,-]$, Dyer–Lashof operations $Q^i_\ell(-)$ for all primes ℓ, ...

Refined data

g \ d	0	1	2	3	4	5	6
2			$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1		$\mathbb{Z}\{\tau\}$	$\mathbb{Z}/10$				
0	$\mathbb{Z}$	$\mathbb{Z}\{\sigma\}$	$\mathbb{Z}\{\sigma^2\}$	$\mathbb{Z}\{\sigma^3\}$	$\mathbb{Z}\{\sigma^4\}$	$\mathbb{Z}\{\sigma^5\}$	$\mathbb{Z}\{\sigma^6\}$

d/g

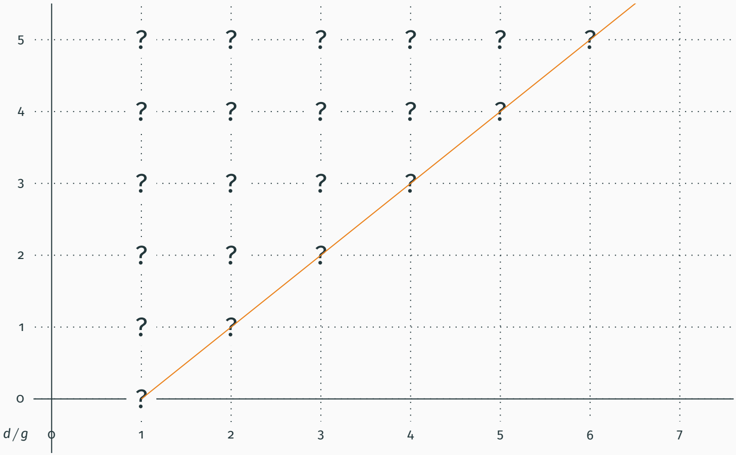
Here τ is the class of a right-handed Dehn twist.

$H_{2,1}(\mathbf{R}^+) = \mathbb{Z}/10$ generated by $\sigma\tau$. Have $[\sigma,\sigma] = 4\sigma\tau$, $Q^1_{\mathbb{Z}}(\sigma) = 3\sigma\tau$.

(For an integral lift $Q^1_{\mathbb{Z}} : H_{*,0}(\mathbf{R}^+) \to H_{*,1}(\mathbf{R}^+)$ of the $\mathbb{F}_2$ Dyer–Lashof operation Q^1_2, defined by universal example.)

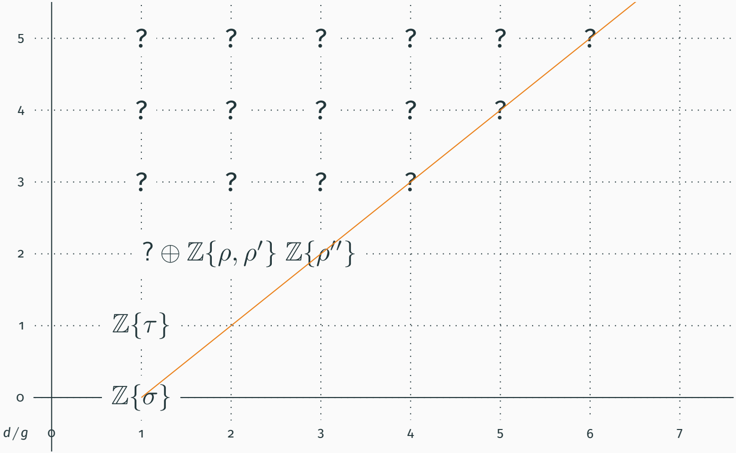
$$\begin{array}{ccccccc}
H_{2,2}(\mathbf{R}^+) & \xrightarrow{-\cdot\sigma} & H_{3,2}(\mathbf{R}^+) & \longrightarrow & H_{3,2}(\mathbf{R}^+/\sigma) & \xrightarrow{\partial} & H_{2,1}(\mathbf{R}^+) \\
\| & & \| & & \| & & \| \\
\mathbb{Z}/2 & \xrightarrow{inj} & \mathbb{Z}\{\lambda\}\oplus\mathbb{Z}/2 & \xrightarrow{\lambda\mapsto 10\mu} & \mathbb{Z}\{\mu\} & \xrightarrow{\mu\mapsto\sigma\tau} & \mathbb{Z}/10\{\sigma\tau\}
\end{array}$$

The vanishing line gives the following chart for $H^{E_2}_{g,d}(\mathbf{R})$.



E_2-homology

Reverse engineering the low-degree E_2-homology lets us complete the chart for $H^{E_2}_{g,d}(\mathbf{R})$ as follows.



Attaching maps are $\partial\rho = 10\sigma\tau$, $\partial\rho' = Q^1_{\mathbb{Z}}(\sigma) - 3\sigma\tau$, $\partial\rho'' = \sigma^2\tau$.

Homological stability

Homological stability

Theorem (Harer, Ivanov, Boldsen, R-W). $H_d(\Gamma_{g,1}, \Gamma_{g-1,1}) = 0$ if $d < \frac{2g}{3}$.

The slope in this statement has been steadily improved, from Harer's original $\frac{1}{3}$ to Ivanov's $\frac{1}{2}$, to the $\frac{2}{3}$ obtained by Boldsen and myself. These proofs were similar in spirit to each other, but all very different to what I present here.

Proof using E_2-cells. Need $H_{g,d}(\mathbf{R}^+/\sigma) = 0$ for $d < \frac{2g}{3}$. Enough to show this with $\mathbb{k}$-coefficients for prime fields $\mathbb{k}$.

Construct a minimal CW-complex model for $\mathbf{R}_\mathbb{k} \in \mathrm{Alg}_{E_2}(\mathrm{sMod}_\mathbb{k}^\mathbb{N})$, a filtered object $f\mathbf{C}$ with $\mathrm{colim}\, f\mathbf{C} \xrightarrow{\sim} \mathbf{R}_\mathbb{k}$. Then

$$\mathrm{gr}(f\mathbf{C}) \simeq \mathbf{E_2}\left(\bigoplus_{\text{cells } \alpha} S_\mathbb{k}^{g_\alpha, d_\alpha, d_\alpha}\right).$$

Can unitalise and form the cofibre $f\mathbf{C}^+/\sigma$ in filtered objects, with

$$\mathrm{gr}(f\mathbf{C}^+/\sigma) \simeq \mathbf{E_2^+}\left(\bigoplus_{\text{cells } \alpha} S_\mathbb{k}^{g_\alpha, d_\alpha, d_\alpha}\right)/\sigma.$$

Proof of homological stability

The spectral sequence for a filtered object is

$$E^1_{*,*,*} = H_{*,*,*}\left(\mathbf{E}_2^+\left(\bigoplus_{\text{cells }\alpha} S_{\Bbbk}^{g_\alpha,d_\alpha,d_\alpha}\right)/\sigma\right) \Rightarrow H_{*,*}(\mathbf{R}_{\Bbbk}^+/\sigma).$$

We now use F. Cohen's calculation of homology of free E_k-algebras.

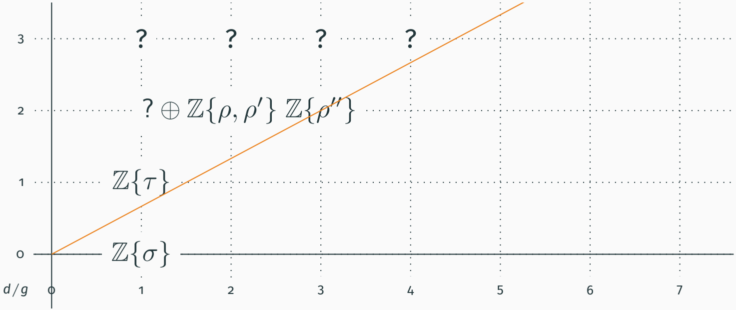
This describes

$$H_{*,*,*}\left(\mathbf{E}_2^+\left(\bigoplus_{\text{cells }\alpha} S_{\mathbb{F}_\ell}^{g_\alpha,d_\alpha,d_\alpha}\right)\right)$$

as given by the free graded-commutative algebra on $Q^I_\ell y$ where y is a basic Lie word in classes x_α of tridegree $(g_\alpha, d_\alpha, d_\alpha)$ and Q^I_ℓ is a Dyer–Lashof monomial satisfying the usual admissibility and excess conditions.

Taking the cofibre of σ simply quotients this by the ideal (σ).

Proof of homological stability



By observation all commutative algebra generators apart from σ, $[\sigma, \sigma]$ if $\ell \neq 2$, and $Q_2^1(\sigma)$ if $\ell = 2$, lie in bidegrees (g, d) with $\frac{d}{g} > \frac{2}{3}$.

Have $Q^1_{\mathbb{Z}}(\sigma) \equiv Q^1_2(\sigma) \mod 2$, and $Q^1_{\mathbb{Z}}(\sigma) \equiv -\frac{1}{2}[\sigma, \sigma] \mod \ell$ for $\ell \neq 2$.

But $d^1(\rho') = Q^1_{\mathbb{Z}}(\sigma) - 3\sigma\tau \equiv Q^1_{\mathbb{Z}}(\sigma) \mod (\sigma)$.

Quotienting out σ and calculating with this, one shows that $E^2_{g,d,*} = 0$ as long as $\frac{d}{g} > \frac{2}{3}$, and so $H_{g,d}(\mathbf{R}^+_{\Bbbk}/\sigma) = 0$ in this range too. □

Secondary homological stability

Secondary homological stability

By thinking about such E_2-cell structures, we discovered the following higher order form of homological stability.

Theorem (Galatius–Kupers–R-W). There are maps

$$\varphi_* \colon H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}; \mathbb{k}) \longrightarrow H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{k})$$

which are epimorphisms for $d \leq \frac{3g-1}{4}$ and isomorphisms for $d \leq \frac{3g-5}{4}$.

(There is also an improved slope for $\mathbb{k} = \mathbb{Q}$, and an extension to surfaces with further marked points and boundaries, and to certain systems of local coefficients. I will not discuss this.)

I will outline the proof of this statement in the case $\mathbb{k} = \mathbb{Q}$, and at the end discuss what needs to be done to promote it to an integral statement.

The rational secondary stabilisation map

We work with $\mathbf{R}^+_{\mathbb{Q}} \in \mathrm{Alg}_{E_2}(\mathrm{sMod}^{\mathbb{N}}_{\mathbb{Q}})$.

As $\mathbf{R}^+_{\mathbb{Q}}$ is an E_2-algebra, the cofibre $\mathbf{R}^+_{\mathbb{Q}}/\sigma$ of right multiplication by σ still has a right $\mathbf{R}^+_{\mathbb{Q}}$-module structure. Represent the class $\lambda \in H_{3,2}(\mathbf{R}^+_{\mathbb{Q}})$ by a map of simplicial modules $\lambda : S^{3,2} \to \mathbf{R}^+_{\mathbb{Q}}$.

Can then form a cofibre sequence

$$\mathbf{R}^+_{\mathbb{Q}}/\sigma \otimes S^{3,2} \xrightarrow{-\cdot\lambda} \mathbf{R}^+_{\mathbb{Q}}/\sigma \longrightarrow \mathbf{R}^+_{\mathbb{Q}}/(\sigma, \lambda).$$

This defines the secondary stabilisation map

$$- \cdot \lambda : H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}; \mathbb{Q}) \longrightarrow H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$$

so need to show that $H_{g,d}(\mathbf{R}^+_{\mathbb{Q}}/(\sigma, \lambda)) = 0$ for $d < \frac{3g}{4}$.

Proof of rational secondary homological stability

Recall the attaching maps for the low-dimensional E_2-cells of $\mathbf{R}^+$ are

$$\partial\rho = 10\sigma\tau, \quad \partial\rho' = Q^1_{\mathbb{Z}}(\sigma) - 3\sigma\tau, \quad \partial\rho'' = \sigma^2\tau$$

and that over $\mathbb{Q}$ we have $Q^1_{\mathbb{Z}}(\sigma) = -\frac{1}{2}[\sigma, \sigma]$.

In particular

$$\partial(-\tfrac{1}{5}(10\rho' + 3\rho)) = [\sigma, \sigma],$$
$$\partial(10\rho'' - \sigma\rho) = 0.$$

In fact the cycle $10\rho'' - \sigma\rho$ represents the class $\lambda \in H_{3,2}(\mathbf{R}^+_{\mathbb{Q}})$.

The above data provides an E_2-map

$$\mathbf{A}^+ := \mathbf{E}_2^+(S^{1,0}\sigma \oplus S^{3,2}\lambda) \cup^{E_2}_{[\sigma,\sigma]} D^{2,2}\rho''' \longrightarrow \mathbf{R}^+_{\mathbb{Q}}.$$

More or less tautologically we have $H^{E_2}_{g,d}(\mathbf{R}_{\mathbb{Q}}, \mathbf{A}) = 0$ for $\frac{d}{g} < \frac{3}{4}$.

Thus $\mathbf{R}^+_{\mathbb{Q}}$ can be obtained from $\mathbf{A}^+$ by attaching (g, d)-cells with $\frac{d}{g} \geq \frac{3}{4}$.

Proof of rational secondary homological stability

If we filter such a relative CW structure

$$\mathbf{A}^+ \longrightarrow \operatorname{colim} f\mathbf{C}^+ \xrightarrow{\sim} \mathbf{R}^+_{\mathbb{Q}}$$

by relative skeleta, it has associated graded

$$\operatorname{gr}(f\mathbf{C}^+) \simeq \mathbf{A}^+ \bigoplus^{E_2} \mathbf{E}_2^+ \left(\bigoplus_{\text{cells } \alpha} S^{g_\alpha, d_\alpha, d_\alpha} \right)$$

with $\frac{d_\alpha}{g_\alpha} \geq \frac{3}{4}$.

Taking the cofibre of $- \cdot \sigma$ and then of $- \cdot \lambda$ in filtered objects, we get a spectral sequence

$$E^1_{*,*,*} = H_{*,*,*}\left(\mathbf{A}^+ \oplus^{E_2} \mathbf{E}_2^+ \left(\oplus_{\text{cells } \alpha} S^{g_\alpha, d_\alpha, d_\alpha}\right)/(\sigma, \lambda)\right) \Rightarrow H_{*,*}(\mathbf{R}^+_{\mathbb{Q}}/(\sigma, \lambda)).$$

We want to show that the target vanishes in slope $< \frac{3}{4}$, and the cells α all have slope $\geq \frac{3}{4}$. It is enough to establish the required vanishing of

$$H_{*,*}(\mathbf{A}^+/(\sigma, \lambda)).$$

Proof of rational secondary homological stability

To do this we give

$$\mathbf{A}^+ = \mathbf{E}_2^+(S^{1,0}\sigma \oplus S^{3,2}\lambda) \cup^{E_2}_{[\sigma,\sigma]} D^{2,2}\rho'''$$

the filtration where σ and λ have filtration 0 and ρ''' has filtration 1. Get a new spectral sequence

$$E^1_{*,*,*} = H_{*,*,*}(\mathbf{E}_2^+(S^{1,0,0}\sigma \oplus S^{3,2,0}\lambda \oplus S^{2,2,1}\rho''')/(\sigma,\lambda)) \Rightarrow H_{*,*}(\mathbf{A}^+/(\sigma,\lambda)),$$

with $d^1(\rho''') = [\sigma,\sigma]$.

By F. Cohen's calculations, $H_{*,*,*}(\mathbf{E}_2^+(S^{1,0,0}\sigma \oplus S^{3,2,0}\lambda \oplus S^{2,2,1}\rho'''))$ is the free graded commutative algebra on the free graded Lie algebra on $\{\sigma,\lambda,\rho'''\}$. The only commutative algebra generators of slope $< \frac{3}{4}$ are $\sigma, \lambda, [\sigma,\sigma]$. The first two are killed by quotienting out by (σ,λ), and the last is $d^1(\rho''')$. A bit of care shows $E^2_{g,d,*} = 0$ for $\frac{d}{g} < \frac{3}{4}$. □

What we know about $H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$

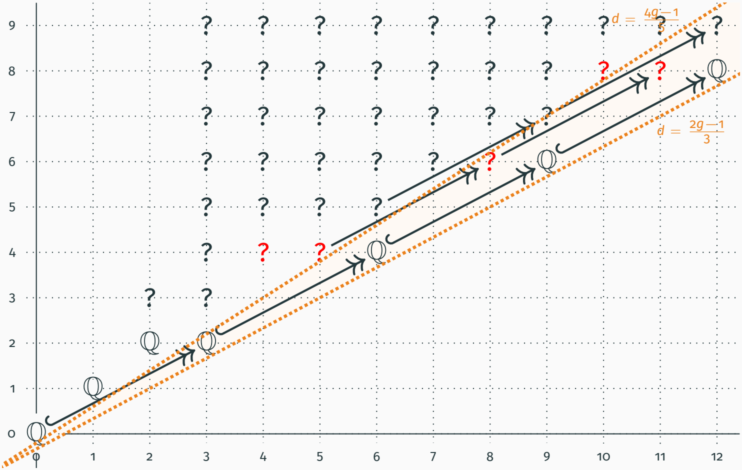


Figure 1: $H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$; ? means unknown, ? means not zero.

Non-zero groups: use results of Faber, Kontsevich, Morita.

Integral secondary homological stability

The secondary homological stability map

The main issue with proving the secondary homological stability theorem with $\mathbb{Z}$-coefficients is formulating what the map should be.

$$\begin{array}{ccccccc}
H_{-1,0}(\mathbf{R}^+) & \xrightarrow{-\cdot\sigma} & H_{0,0}(\mathbf{R}^+) & \xrightarrow{\sim} & H_{0,0}(\mathbf{R}^+/\sigma) & \xrightarrow{\partial} & H_{-1,-1}(\mathbf{R}^+) \\
\downarrow & & \downarrow{-\cdot\lambda} & & \downarrow{-\cdot\lambda} & & \downarrow{-\cdot\lambda} \\
H_{2,2}(\mathbf{R}^+) & \xrightarrow{-\cdot\sigma} & H_{3,2}(\mathbf{R}^+) & \longrightarrow & H_{3,2}(\mathbf{R}^+/\sigma) & \xrightarrow{\partial} & H_{2,1}(\mathbf{R}^+) \\
\| & & \| & & \| & & \| \\
\mathbb{Z}/2 & \xrightarrow{inj} & \mathbb{Z}\{\lambda\} \oplus \mathbb{Z}/2 & \xrightarrow{\lambda \mapsto 10\mu} & \mathbb{Z}\{\mu\} & \xrightarrow{\mu \mapsto \sigma\tau} & \mathbb{Z}/10\{\sigma\tau\}
\end{array}$$

shows that

$$- \cdot \lambda : \mathbb{Z} = H_{0,0}(\mathbf{R}^+/\sigma) \longrightarrow H_{3,2}(\mathbf{R}^+/\sigma) = \mathbb{Z}$$

is multiplication by 10, so not surjective: thus this map cannot induce an epi/isomorphism in the desired range.

Instead want to "multiply by μ", but μ is not a class on $\mathbf{R}^+$, only on $\mathbf{R}^+/\sigma$, which is not an E_2-algebra…

The secondary homological stability map

To resolve this extend $\mu : S^{3,2} \to \mathbf{R}^+/\sigma$ to a right $\mathbf{R}^+$-module map

$$\mu^{ad} : S^{3,2} \otimes \mathbf{R}^+ \longrightarrow \mathbf{R}^+/\sigma$$

and observe that the right $\mathbf{R}^+$-module map

$$S^{3,2} \otimes S^{1,0} \otimes \mathbf{R}^+ \xrightarrow{S^{3,2}\otimes\sigma} S^{3,2} \otimes \mathbf{R}^+ \longrightarrow \mathbf{R}^+/\sigma$$

corresponds to a class in the group $H_{4,2}(\mathbf{R}^+/\sigma)$, which vanishes by ordinary homological stability. Choosing a nullhomotopy gives an extension

$$\varphi : S^{3,2} \otimes \mathbf{R}^+/\sigma \longrightarrow \mathbf{R}^+/\sigma,$$

which is a secondary stabilisation map. Want to show that the cofibre $\mathbf{C}_\varphi$ has a slope $\frac{3}{4}$ vanishing line.

In fact get $H_{4,3}(\mathbf{R}^+/\sigma)$-many such maps (don't know what this group is), but will show all their cofibres have the required vanishing.

Proof of integral secondary homological stability

The first step is to show that *all* the secondary stabilisation maps

$$\varphi : S^{3,2} \otimes \mathbf{R}^{+}/\sigma \longrightarrow \mathbf{R}^{+}/\sigma$$

just constructed may be promoted to filtered maps, where $\mathbf{R}^{+}$ is given its skeletal filtration for a minimal CW-structure, and $S^{3,2}$ is given filtration precisely 3.

This means repeating the obstruction-theory argument which constructed φ but now in the category of filtered objects.

The groups carrying the obstruction (and parameterising choices of nullhomotopy if the obstruction vanishes) in the category of filtered objects are in principle different, and it requires some slightly subtle work to show that the φ can indeed all be promoted to filtered maps.

Proof of integral secondary homological stability

Now that the φ are lifted to filtered maps, we get a filtration on their cofibres $\mathbf{C}_\varphi$. To establish $H_{g,d}(\mathbf{C}_\varphi) = 0$ for $\frac{d}{g} < \frac{3}{4}$ it is enough to do so with $\mathbb{F}_\ell$-coefficients for each prime ℓ.

The filtration gives a spectral sequence with E^1-page

$$H_{*,*,*}\left(\left(S^{0,0,0}_{\mathbb{F}_\ell} \oplus S^{3,3,3}_{\mathbb{F}_\ell}\kappa\right) \otimes \mathbf{E}_2^+\left(S^{1,0,0}_{\mathbb{F}_\ell}\sigma \oplus S^{1,1,1}_{\mathbb{F}_\ell}\tau \oplus S^{2,2,2}_{\mathbb{F}_\ell}\rho \oplus S^{2,2,2}_{\mathbb{F}_\ell}\rho' \oplus S^{3,2,2}_{\mathbb{F}_\ell}\rho'' \oplus \bigoplus_\alpha S^{g_\alpha,d_\alpha,d_\alpha}_{\mathbb{F}_\ell}\right)/\sigma\right)$$

converging to $H_{*,*}(\mathbf{C}_\varphi; \mathbb{F}_\ell)$. This has

$$d^1(\kappa) = \rho'', \quad d^1(\rho') = Q^1_{\mathbb{Z}}(\sigma) = \begin{cases} Q^1_2(\sigma) & \ell = 2 \\ -\frac{1}{2}[\sigma,\sigma] & \ell \neq 2 \end{cases}$$

and $\sigma, \tau, \rho, \rho''$ are d^1-cycles. Using this and F. Cohen's description of the homology of free E_2-algebras, some homological algebra shows that $E^2_{g,p,q} = 0$ for $\frac{p+q}{g} < \frac{3}{4}$, and hence $H_{g,d}(\mathbf{C}_\varphi; \mathbb{F}_\ell) = 0$ for $\frac{d}{g} < \frac{3}{4}$. $\square$

Literature

Based on work with S. Galatius and A. Kupers:

Cellular E_k-algebras.
arXiv:1805.07184.

E_2-cells and mapping class groups.
Publ. Math. Inst. Hautes Études Sci. 130 (2019), 1–61.

For further applications of these ideas see also:

E_∞-cells and general linear groups of finite fields.
arXiv:1810.11931.

E_∞-cells and general linear groups of infinite fields.
Forthcoming.