The space of surfaces in a manifold

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joint work with Federico Cantero

Surfaces in a manifold

Let M be a smooth manifold of dimension d, possibly with boundary ∂M.

Definition

Let $\mathcal{E}(M)$ denote the set of pairs (X, ℓ_X), where $X \subset M$ is a subset which has the structure of a compact, closed, connected, smooth, 2-dimensional submanifold, and ℓ_X is an orientation of X.

Choosing for each integer $g \geq 0$ an oriented surface Σ_g of genus g, there is a surjective function

$$\coprod_{g \geq 0} \mathrm{Emb}(\Sigma_g, M) \longrightarrow \mathcal{E}(M),$$

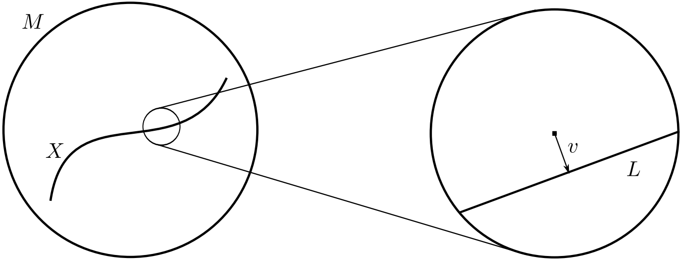
given by sending an embedding to its image. We equip $\mathrm{Emb}(\Sigma_g, M)$ with the Whitney topology, and $\mathcal{E}(M)$ with the quotient topology.

We denote by $\mathcal{E}_g(M)$ the path component which is the quotient of $\mathrm{Emb}(\Sigma_g, M)$. Our aim is to study the topology of the space $\mathcal{E}_g(M)$, and in particular the cohomology $H^*(\mathcal{E}_g(M))$.

The philosophy of scanning

The space $\mathcal{E}(M)$ has points consisting of oriented, closed surfaces in M. What does this look like "at the microscopic scale" in M?

At a small scale, M looks like a d-dimensional vector space V, and an oriented surface looks like an oriented affine plane $\mathbb{R}^2 \subset V$. Or, if we are far away from the surface, it looks like $\varnothing \subset V$.



That is, we locally see an oriented 2-plane in V, and a normal vector. This normal vector can also be "infinitely long", and we then forget about the oriented 2-plane.

The philosophy of scanning (cont.)

The space of oriented affine planes in V containing the origin is the Grassmannian $\mathrm{Gr}_2^+(V)$. If we do not insist that the plane contains the origin, then an oriented affine plane is given by a pair

$$(L \in \mathrm{Gr}_2^+(V), v \in L^\perp),$$

so the space of them is the total space of the vector bundle $\gamma_2^\perp \to \mathrm{Gr}_2^+(V)$. If the vector v gets very long, then the affine plane is far away from the origin: thus, adding the point $\varnothing$ gives the Thom space

$$\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(V)).$$

Definition

We call this $\mathcal{S}(V)$. It is natural in the vector space V.

We think of $\mathcal{S}(V)$ as *the space of (possibly empty) affine surfaces in V*.

The scanning map

To describe scanning honestly, we have to modify the space $\mathcal{E}(M)$ slightly. Choose a Riemannian metric g on M, and let

$$\mathcal{E}^\nu(M) \subset (0,\infty) \times \mathcal{E}(M)$$

be the set of pairs (ε, X) such that the exponential map $\exp : \nu(X) \to M$ is an embedding when restricted to vectors of length at most ε.
We define a map

$$M \times \mathcal{E}^\nu(M) \longrightarrow \mathcal{S}^{fib}(TM) = \sqcup_{m \in M} \mathcal{S}(T_m M)$$
$$(x, \varepsilon, X) \longmapsto \begin{cases} \varnothing \in \mathcal{S}(T_x M) & x \notin \nu_\varepsilon(X) \\ (T_p X + v \subset T_x M, \frac{v}{\varepsilon - |v|} \in (T_p X)^\perp) & x = v \in \nu_\varepsilon(X)_p \end{cases}$$

with adjoint the *scanning map*

$$\mathcal{E}^\nu(M) \longrightarrow \Gamma_c(\mathcal{S}^{fib}(TM) \to M).$$

The main theorem

If M is simply-connected and of dimension at least 6, one can show that there is a natural bijection

$$\pi_0(\Gamma_c(\mathcal{S}^{fib}(TM) \to M)) \cong \mathbb{Z} \times H_2(M;\mathbb{Z})$$

under which the scanning map sends (X, ℓ_X) to $(\frac{\chi(X)}{2}, [X])$. We let $\Gamma_c(\mathcal{S}^{fib}(TM) \to M)_g$ denote those path components corresponding to $\{1-g\} \times H_2(M;\mathbb{Z})$.

Theorem (Cantero–R-W)

If M is simply-connected and of dimension at least 6, the scanning map

$$\mathcal{E}_g(M) \longrightarrow \Gamma_c(\mathcal{S}^{fib}(TM) \to M)_g$$

induces an isomorphism on integral homology in degrees $* \leq \frac{2g-2}{3}$.

Relation to previous work

This result is based on, and recovers, several older results. If we let $M = \mathbb{R}^n$, we obtain the statement that

$$\mathcal{E}_g(\mathbb{R}^n) \longrightarrow \Omega^n \mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(\mathbb{R}^n))$$

is an isomorphism on integral homology in degrees $* \leq \frac{2g-2}{3}$.
Taking the limit as $n \to \infty$,

- $\mathcal{E}_g(\mathbb{R}^n)$ approximates $B\mathrm{Diff}^+(\Sigma_g)$, the classifying space of the diffeomorphism group of Σ_g,
- the right-hand side becomes the infinite loop space of a certain spectrum (in the sense of stable homotopy theory) $\mathbf{MTSO}(2)$.

Theorem (Madsen–Weiss)

There is a map

$$B\mathrm{Diff}^+(\Sigma_g) \longrightarrow \Omega^\infty_\bullet \mathbf{MTSO}(2)$$

which induces an isomorphism on integral homology in degrees $* \leq \frac{2g-2}{3}$.

Relation to previous work (cont.)

Theorem (Madsen–Weiss)

There is a map

$$B\mathrm{Diff}^+(\Sigma_g) \longrightarrow \Omega^\infty_\bullet \mathbf{MTSO}(2)$$

which induces an isomorphism on integral homology in degrees $* \leq \frac{2g-2}{3}$.

The right-hand side is independent of g, so this formulation of the Madsen–Weiss theorem also implies that $H_*(B\mathrm{Diff}^+(\Sigma_g); \mathbb{Z})$ is independent of g in the stable range. This was an older result, due to Harer, with improvements to the stable range by Ivanov, Boldsen, and R-W.

Because there is no direct way to compare $B\mathrm{Diff}^+(\Sigma_g)$ and $B\mathrm{Diff}^+(\Sigma_{g+1})$, to prove Harer's stability theorem one must work with *surfaces with boundary*; we must do the same.

Surfaces with boundary

If M is a manifold with boundary, and we are given a 1-manifold $\delta \subset M$ consisting of b circles, define

$$\mathcal{E}(M; \delta)$$

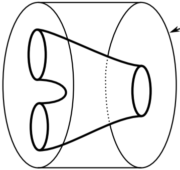
to be the set of pairs (X, ℓ_X) where $X \subset M$ is a connected, smooth, 2-dimensional submanifold with boundary $\delta \subset \partial M$, and ℓ_X is an orientation.

We give this a topology as before, and let $\mathcal{E}_{g,b}(M; \delta)$ denote those path components where the surface has genus g (and b boundary components).

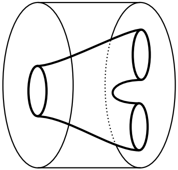
If $b = 0$, there is only one possible δ and this recovers our old definition.

Stabilisation maps

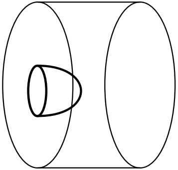
$[0,1] \times \partial M$



$\alpha_{g,b}$ $: \mathcal{E}_{g,b}(M;\delta) \longrightarrow \mathcal{E}_{g+1,b-1}(M;\delta')$



$\beta_{g,b}$ $: \mathcal{E}_{g,b}(M;\delta) \longrightarrow \mathcal{E}_{g,b+1}(M;\delta'')$



$\gamma_{g,b}$ $: \mathcal{E}_{g,b}(M;\delta) \longrightarrow \mathcal{E}_{g,b-1}(M;\delta''')$

Homological stability

Theorem (Cantero–R-W)

Let M be simply connected and of dimension at least 6.

1. *Any map $\alpha_{g,b}$ induces an isomorphism in homology in degrees $* \leq \frac{2g-2}{3}$ and an epimorphism in degrees $* \leq \frac{2g+1}{3}$.*
2. *Any map $\beta_{g,b}$ induces an isomorphism in homology in degrees $* \leq \frac{2g-3}{3}$ and an epimorphism in degrees $* \leq \frac{2g}{3}$. (If one of the outgoing boundary conditions on the pair of pants is contractible in ∂M then the map $\beta_{g,b}$ is also a monomorphism in all degrees.)*
3. *Any map $\gamma_{g,b}$ induces an isomorphism in homology in degrees $* \leq \frac{2g}{3}$ and an epimorphism in degrees $* \leq \frac{2g+3}{3}$. If $b \geq 2$, then it is always an epimorphism.*

The proof of this theorem is the most delicate and technically involved of any stability theorem I have seen.

Identifying the stable homology (outline)

To identify the stable homology as that of $\Gamma_c(\mathcal{S}^{fib}(TM) \to M)$, we use a modification of an argument of Galatius, Madsen, Tillmann, and Weiss. Let $U \subset \partial M$ be a ball, and $W = M \cup_U (U \times [0, \infty))$.

- A "group completion" result, which relates the space $\mathcal{E}(M; \delta)$ to the fibre of a map
$$Bp : B(\mathcal{C}_2^\partial(U) \wr F) \longrightarrow B\mathcal{C}_2^\partial(U) \tag{1}$$
between classifying spaces of certain cobordism categories,
- A "parametrised surgery" result, which identifies the classifying spaces of the categories $\mathcal{C}_2^\partial(U)$ and $\mathcal{C}_2^\partial(U) \wr F$ to those of less specialised categories $\mathcal{C}_2(U)$ and $\mathcal{C}_2(U) \wr F$,
- An "h-principle" result, which identifies the classifying space of $\mathcal{C}_2(U)$ and $\mathcal{C}_2(U) \wr F$ with spaces of sections.
- In total, this identifies (1) with the fibration
$$\Gamma(\mathcal{S}^{fib}(TW) \to W) \longrightarrow \Gamma(\mathcal{S}^{fib}(TU \times (0, \infty)) \to U \times [0, \infty))$$
whose fibre is $\Gamma_c(\mathcal{S}^{fib}(TM) \to M)$.

Identifying the stable homology (detail)

Define a category $\mathcal{C}_2(U)$:

- Objects are: a real number $t \in \mathbb{R}_{>0}$ and a 1-manifold $\delta \subset \{t\} \times U$.
- Morphisms are: a pairs of real numbers $t_0 < t_1 \in \mathbb{R}_{>0}$ and a surface $X \subset [t_0, t_1] \times U$, which is collared near the boundary.
- Source and target maps are given by intersecting X with $\{t_0\} \times U$ and $\{t_1\} \times U$ respectively; composition is given by concatenation of cobordisms.
- The objects and morphisms of this category are suitably topologised.

Let $M_t = M \cup_U (U \times [0, t])$. We define a functor

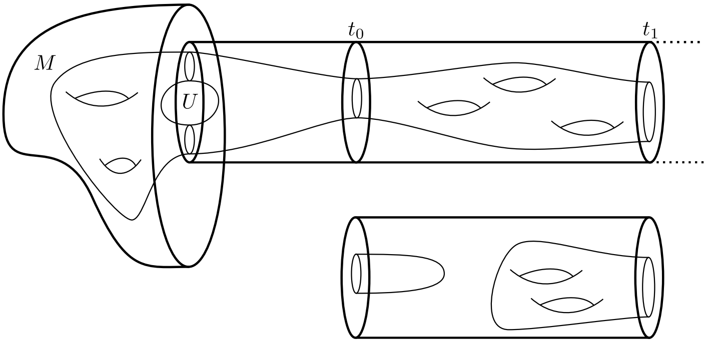
$$F : \mathcal{C}_2(U) \longrightarrow \textbf{Top}$$

by sending $\delta \subset \{t\} \times U$ to the space $\mathcal{E}(M_t; \delta)$, and given a cobordism $X \subset [t, s] \times U$ from δ to δ', gluing it on gives a map

$$\mathcal{E}(M_t; \delta) \longrightarrow \mathcal{E}(M_s; \delta').$$

Identifying the stable homology (detail, cont.)

This in not quite true: the space $\mathcal{E}(M_t;\delta)$ consists only of *connected* surfaces, whereas morphisms in $\mathcal{C}_2(U)$ do not have to be connected. To get a functor, we must restrict to the subcategory $\mathcal{C}_2^{\partial}(U)$ where we only allow those morphisms $X \subset [t_0, t_1] \times U$ such that the pair $(X, X \cap \{t_0\} \times U)$ is connected.



Identifying the stable homology (detail, cont.)

We can form the Grothendieck construction

$$p : \mathcal{C}_2^\partial(U) \wr F \longrightarrow \mathcal{C}_2^\partial(U) \quad \Rightarrow \quad Bp : B(\mathcal{C}_2^\partial(U) \wr F) \longrightarrow B\mathcal{C}_2^\partial(U).$$

The fibre of this map (over a suitable basepoint) is $\mathcal{E}(M;\delta)$.

A cobordism $X : \delta \rightsquigarrow \delta$ gives a loop in $B\mathcal{C}_2^\partial(U)$, and this induces the map

$$- \circ X : \mathcal{E}(M;\delta) \to \mathcal{E}(M;\delta)$$

on the fibre. *This is not an equivalence: it will not even typically be surjective on* π_0*.* Thus Bp is not a fibration.

By suitably stabilising the functor F, we can make Bp be a *homology fibration*: its fibre is homology equivalent to its homotopy fibre. This is why the main theorem is homological, not homotopical; it is unavoidable.

Bibliography

For more details see

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