# Homeomorphisms of $\mathbb{R}^d$

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European Research Council
Established by the European Commission

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I will try to express what we know via the *homotopy groups*

$$\pi_n(BTop(d)) = \pi_{n-1}(Top(d)) = \frac{\{\text{continuous maps } f : S^{n-1} \to Top(d)\}}{\text{homotopy}}$$

or, suppressing torsion, their rationalisations $\pi_n(BTop(d)) \otimes \mathbb{Q}$.

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The Gram–Schmidt process deforms $GL(d)$ to its subgroup $O(d)$.

$O(d)$ is a compact Lie group and its topology is well understood:

e.g. $\pi_*(BO(d)) \otimes \mathbb{Q} = \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ 0 & d \text{ odd.} \end{cases}$

# Stabilising by dimension

There are inclusions

$$\begin{array}{ccccccccc}
O(d) & \longrightarrow & O(d+1) & \longrightarrow & O(d+2) & \longrightarrow & \cdots & \longrightarrow & O \\
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$\pi_n(\frac{Top}{O}) \cong \Theta_n := \{$smooth oriented $n$-manifolds homeomorphic to $S^n\}$,

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The theorem of Kervaire–Milnor determines these groups:

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| $\Theta_5 = 0$ | $\Theta_6 = 0$ | $\Theta_7 = \mathbb{Z}/28$ | $\Theta_8 = \mathbb{Z}/2$ | $\Theta_9 = (\mathbb{Z}/2)^3$ |
| $\Theta_{10} = \mathbb{Z}/6$ | $\Theta_{11} = \mathbb{Z}/992$ | $\Theta_{12} = 0$ | $\Theta_{13} = \mathbb{Z}/3$ | $\Theta_{14} = \mathbb{Z}/2$ |

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$$\Rightarrow \pi_*(BTop) \otimes \mathbb{Q} \cong \pi_*(BO) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i]$$

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**Smooth case.** We have $\frac{O(d+1)}{O(d)} \cong S^d$, as $O(d+1)$ acts transitively on $S^d$ with stabiliser $O(d)$. Thus $O(d) \to O(d+1)$ is $(d-1)$-connected.

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Furthermore, these differences can be related to one another:

$$\frac{O(d+1)}{O(d)} \longrightarrow \Omega \frac{O(d+2)}{O(d+1)} = map_*(S^1, \tfrac{O(d+2)}{O(d+1)})$$
$$O(d) \cdot A \longmapsto (\theta \mapsto O(d+1) \cdot R_\theta (A \oplus 1) R_\theta^{-1}),$$

where $R_\theta \in O(d+2)$ rotates by $\theta$ in the last two coordinates.

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The source and target of this map are both $(d-1)$-connected, but the map is $(2d-1)$-connected: this is *Freudenthal's suspension theorem*.

# Interlude: stable homotopy theory

This data, a collection $\mathbb{X}$ of based spaces $X_d$ and structure maps $X_d \to \Omega X_{d+1}$, is precisely a *spectrum* in the sense of stable homotopy theory. The example here is the *sphere spectrum* $\mathbb{S}$.

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Using the structure maps we can make sense of

$$\pi_i(\mathbb{X}) := \operatorname*{colim}_{d\to\infty} \pi_{i+d}(X_d)$$

and so on. The next basic theorem in this subject is

$$\pi_i(\mathbb{S}) = \begin{cases} 0 & i < 0 \\ \mathbb{Z} & i = 0 \\ \text{finite abelian} & i > 0. \end{cases}$$

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In relation to our story, we have

$$\pi_{i+d}\left(\tfrac{O(d+1)}{O(d)}\right) \cong \pi_i(\mathbb{S})$$

for $i + d < 2d - 1$. This gives a sense in which the homotopy groups of $\frac{O(d+1)}{O(d)}$ are "the same" for varying $d$.

# Stabilising by dimension III

**Topological case.** $\frac{Top(d+1)}{Top(d)}$ cannot be identified with a known space: it is its own thing. It is still $(d-1)$-connected. The same rotation map as before makes the collection of these spaces into a spectrum, though now the structure map

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is only known to be $\sim \frac{4}{3}d$-connected (Igusa '88).

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for $d + * \lesssim \frac{4}{3}d$. In degrees $\lesssim \frac{4}{3}d$ this leads to

$$\pi_*(BTop(d)) \otimes \mathbb{Q} = \bigoplus_{i=1}^{\infty} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ \bigoplus_{j=1}^{\infty} \mathbb{Q}[d+1+4j] & d \text{ odd.} \end{cases}$$

# A pattern

The story so far was complete by 1988, and not much had changed until recently. The impetus has been a ’15 theorem of Weiss on “topological Pontrjagin classes”, and especially a perspective adopted in his argument.

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**Theorem (Kupers–R-W ’20).** $\pi_*(\frac{Top}{Top(2n)}) \otimes \mathbb{Q}$ is $\mathbb{Q}[2n]$ plus classes in the bands of degrees

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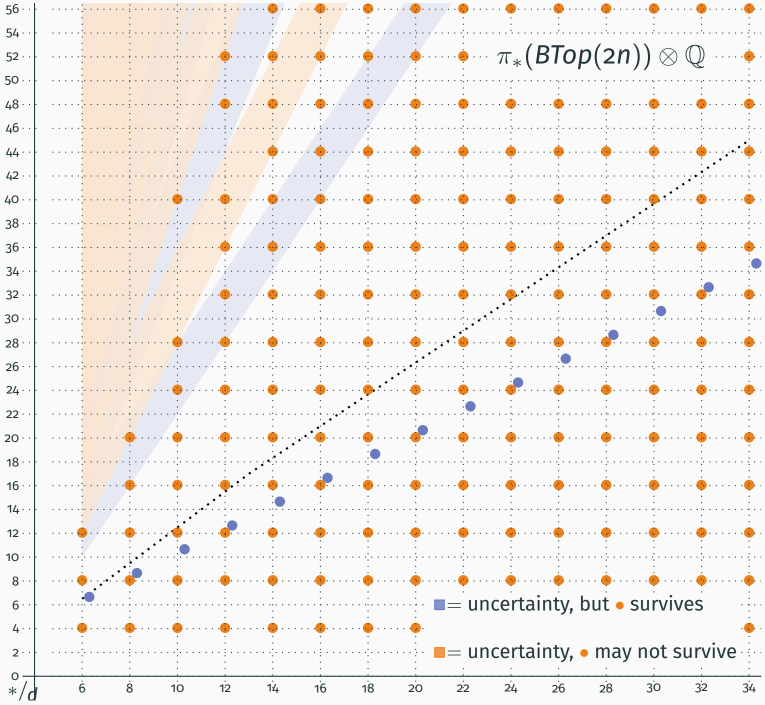
Furthermore, there is something nontrivial in the $s = 3$ band:

$$\mathbb{Q}^2 \longleftarrow \mathbb{Q}^4 \longleftarrow \mathbb{Q}^{10} \longleftarrow \mathbb{Q}^{21} \longleftarrow \mathbb{Q}^{15} \hookleftarrow \mathbb{Q}^3$$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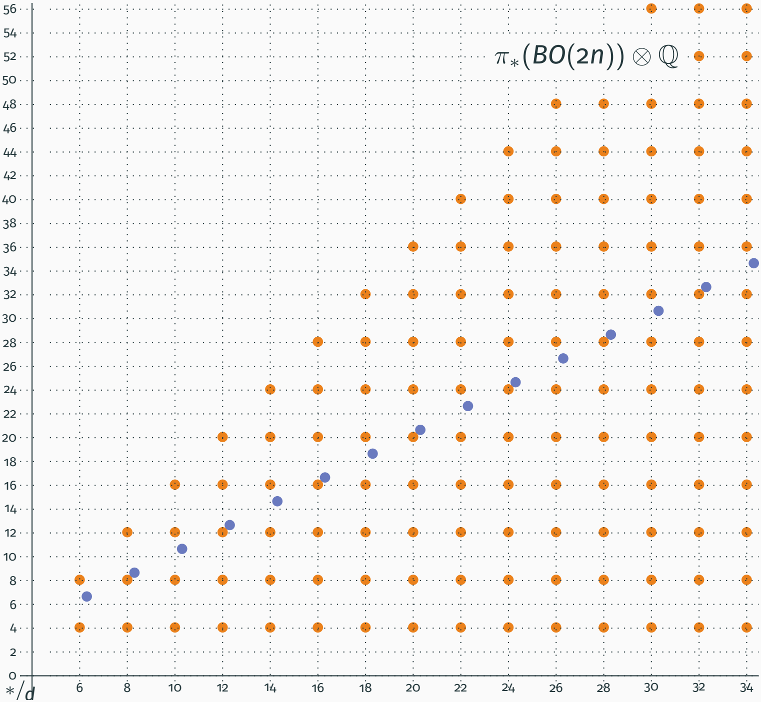
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$$\frac{Homeo_\partial(D^d)}{Diff_\partial(D^d)} \simeq \Omega^d_0\left(\frac{Homeo(\mathbb{R}^d)}{Diff(\mathbb{R}^d)}\right) \simeq \Omega^d_0\left(\frac{Top(d)}{O(d)}\right).$$

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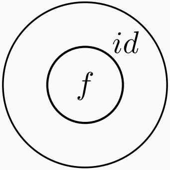


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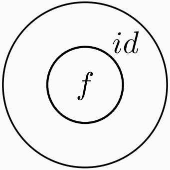


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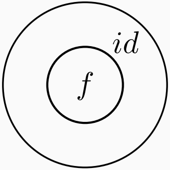


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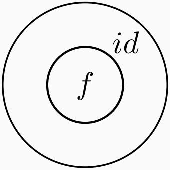


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So understanding homeomorphisms of $\mathbb{R}^d$ is more or less the same as understanding diffeomorphisms of $D^d$, and this is how it is usually approached.

# Indications on the proof II

## Stabilising by complexity

A programme of Galatius and myself, extending the Madsen–Weiss theorem to high dimensions, gives a good understanding of diffeomorphism groups of manifolds of dimension $2n$ which are "complicated" in the sense that they contain many $S^n \times S^n$'s.

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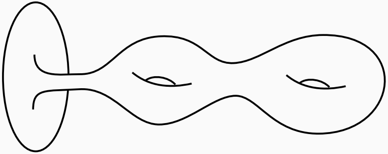

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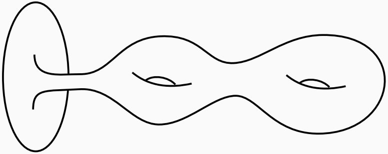

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**Theorem. (Madsen–Weiss '07 $2n = 2$, Galatius–R-W '14 $2n \geq 4$)**

$$\lim_{g \to \infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$$

Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

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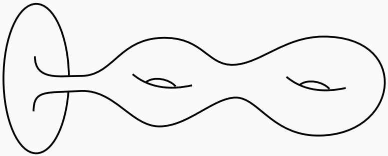

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Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

(For $2n \neq 4$ there is also a "stability theorem" saying how quickly the limit is attained.)

# Indications on the proof III

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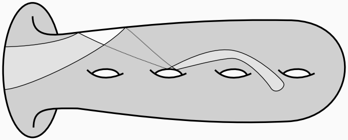

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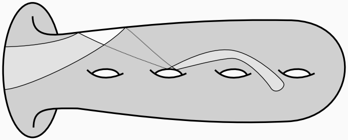

Because of the change of boundary conditions, these embeddings have "codimension $n$" from the point of view of embedding theory. If $n \geq 3$ this space is therefore accessible using the Goodwillie–Weiss "calculus of embeddings".