Spaces of manifolds

Oscar Randal-Williams

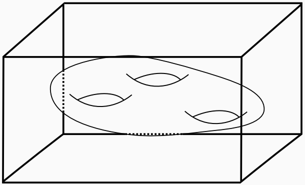
European Research Council
Established by the European Commission

LEVERHULME TRUST

What are spaces of manifolds?

Manifolds in euclidean space

$$\mathcal{M}_d = \left\{ X \subset \mathbb{R}^\infty \,\middle|\, \begin{array}{c} X \text{ is a smooth, compact,} \\ \text{closed, } d\text{-dimensional} \\ \text{submanifold} \end{array} \right\}$$



For such a manifold W have

$$\begin{aligned} \mathrm{Emb}(W, \mathbb{R}^\infty) &\longrightarrow \mathcal{M}_d \\ e &\longmapsto e(W) \end{aligned}$$

and we give $\mathcal{M}_d$ the finest topology making all of these maps continuous, when the domain has the Whitney C^∞-topology.

In particular we have the subspaces

$$\mathcal{M}(W) = \{X \in \mathcal{M}_d \,|\, X \text{ is diffeomorphic to } W\}$$

Diffeomorphism groups and fibre bundles

We have

$$\begin{aligned}\mathcal{M}(W) &= \mathit{Emb}(W, \mathbb{R}^\infty)/\mathit{Diff}(W)\\ &= \{\text{a contractible space on which } \mathit{Diff}(W) \text{ acts freely}\}/\mathit{Diff}(W)\\ &=: \mathit{BDiff}(W),\end{aligned}$$

a model for the *classifying space* of the topological group $\mathit{Diff}(W)$.

Form the tautological space

$$\mathcal{E}(W) := \{(X, x) \in \mathcal{M}(W) \times \mathbb{R}^\infty \,|\, x \in X\}$$

Theorem. The forgetful map $\pi : \mathcal{E}(W) \to \mathcal{M}(W)$ is the universal smooth W-bundle, i.e. the map

$$\begin{aligned}\left\{\begin{matrix}\text{maps } f : B \to \mathcal{M}(W),\\ \text{up to homotopy}\end{matrix}\right\} &\longrightarrow \left\{\begin{matrix}\text{smooth } W\text{-bundles over } B,\\ \text{up to isomorphism}\end{matrix}\right\}\\ f \quad &\longmapsto \quad f^*\pi : f^*\mathcal{E}(W) \to B\end{aligned}$$

is a bijection for any (reasonable) B.

Invariants of $\mathcal{M}(W)$

The standard invariants of algebraic topology, applied to $\mathcal{M}(W)$, have an immediate interpretation:

(i) Homotopy groups $\pi_n(\mathcal{M}(W)) = \left\{\begin{smallmatrix}\text{based maps } S^n \to \mathcal{M}(W) \\ \text{up to homotopy}\end{smallmatrix}\right\}$ tautologically parameterise smooth W-bundles over S^n.

(ii) Such bundles are obtained by gluing trivial bundles over the upper and lower hemisphere of S^n along a map $S^{n-1} \to \mathit{Diff}(W)$ ("clutching"), giving a bijection with $\pi_{n-1}(\mathit{Diff}(W))$.

(iii) In particular

$$\pi_1(\mathcal{M}(W)) \cong \left\{\begin{smallmatrix}\text{diffeomorphisms } \varphi : W \to W, \\ \text{up to isotopy}\end{smallmatrix}\right\}$$

is the *mapping class group* of W.

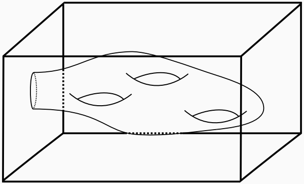
(iv) Cohomology groups $H^n(\mathcal{M}(W))$ tautologically parameterise *characteristic classes* of smooth W-bundles.

Variants

If W is a compact manifold with non-empty boundary ∂W, then we may choose an embedding $\partial W \subset \mathbb{R}^{\infty-1}$ and form a space

$$\mathcal{M}(W) = \left\{ X \subset [0,\infty) \times \mathbb{R}^{\infty-1} \mid \begin{array}{c} X \text{ is a smooth, compact,} \\ d\text{-dimensional submanifold} \\ \text{with boundary } \{0\} \times \partial W, \\ \text{diffeomorphic to } W \text{ relative to } \partial W \end{array} \right\}$$

Topologised similarly. It is a model for $BDiff(W)$, where $Diff(W)$ denotes the group of diffeomorphisms of W fixing the boundary.



Can easily make variants where manifolds are equipped with “tangential structures” such as orientations, Spin structures, framings, and so on.

Why?

Classification of manifolds $= \pi_0(\mathcal{M}_d)$

$\Rightarrow$ categorifies to understanding the homotopy type of $\mathcal{M}_d$

As classifying spaces for smooth fibre bundles the spaces $\mathcal{M}(W) \simeq BDiff(W)$ are central objects in geometric topology, and we are required to investigate their topology.

Fact. We don't know $H^*(\mathcal{M}(W); \mathbb{Q})$ for any W of dimension > 3.

$\Rightarrow$ natural sense of outrage

Diff(W) acts on any natural space of differential-geometric data associated to W (Riemannian metrics with curvature conditions, symplectic forms, ...)

$\Rightarrow$ can be used to probe the topology of such spaces.

Stabilising and scanning

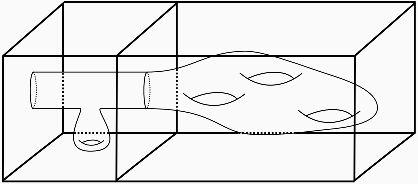
Outline

For manifolds of even dimension $d = 2n$, a powerful technique for understanding the cohomology of $\mathcal{M}(W)$ has arisen over the last 10 years.

It is inspired by Madsen and Weiss' proof of the Mumford conjecture, corresponding to the case $d = 2$.

If W is a $2n$-manifold with non-empty boundary, there are maps

$$\sigma : \mathcal{M}(W) \longrightarrow \mathcal{M}(W \# S^n \times S^n)$$



Let $g(W) := \max\{g \in \mathbb{N} \mid W \cong \#^g S^n \times S^n \# W'\}$ be the *genus* of W.

Theorem. The map σ is an isomorphism on i-th (co)homology for

(i) $i \leq 3g(W)/2$ if $2n = 0$, [Nakaoka '60]
(ii) $i \leq (2g(W) - 2)/3$ if $2n = 2$, [Harer '85, Ivanov '91, Wahl '08, Boldsen '12, R-W '16]
(iii) $i \leq (g(W) - 3)/2$ if $2n \geq 6$ and W is simply-connected [Galatius–R-W '18] or has virtually poly-$\mathbb{Z}$ fundamental group. [Friedrich '17]

It is clarifying to introduce spaces of noncompact manifolds:

$$\psi_d(N) = \{X \subset B_1(0) \subset \mathbb{R}^N \mid {}^{X \text{ is a } d\text{-dimensional submanifold}}_{\text{closed as a topological subspace}}\}$$

This seems complicated, but homotopically is easy: a rescaling argument shows it deforms to its subspace of affine manifolds

$$\{(P,p) \in Gr_d(\mathbb{R}^N) \times \mathbb{R}^N \mid p \in P^\perp\}/\{(P,p) \text{ with } |p| \geq 1\}$$

This is a standard construction in algebraic topology: the Thom space of the orthogonal complement of the tautological bundle over the Grassmannian $Gr_d(\mathbb{R}^N)$.

Scanning

Writing $\mathcal{M}(W)_N = \{X \in \mathcal{M}(W) \,|\, X \subset \mathbb{R}^N\}$, there is a "scanning map"

$$\begin{aligned} \alpha_N : \mathcal{M}(W)_N &\longrightarrow \mathrm{map}_*(S^N, \psi_d(\mathbb{R}^N)) =: \Omega^N \psi_d(\mathbb{R}^N) \\ X &\longmapsto (v \mapsto X \cap B_1(v)) \end{aligned}$$

which we think of as recording, for a given X, the continuously varying collection of all local pictures of X.

Taking the limit as $N \to \infty$ gives a map

$$\alpha : \mathcal{M}(W) \longrightarrow \operatorname*{colim}_{N \to \infty} \Omega^N \psi_d(\mathbb{R}^N) =: \Omega^\infty \Psi_d$$

This can't be a very good approximation, as the right-hand side depends only on the dimension d and not on the specific d-manifold W.

Surprisingly, after a small modification it is a good approximation.

Scanning

To a $2n$-manifold W there is an associated "tangential structure" θ_W. It would take us out of our way to define it here, but as examples:

(i) If W is an orientable surface then θ_W is "an orientation".
(ii) If W is a non-orientable surface then θ_W is "no structure".
(iii) If $W = \mathbb{CP}^2$ then θ_W is "an orientation".
(iv) If $W = S^2 \times S^2$ then θ_W is "a Spin structure".
(v) If $W = \#^g S^n \times S^n$ then θ_W is "a framing on the n-skeleton".

Theorem. For any $2n$-manifold W the map

$$\lim_{g \to \infty} H^*(\mathcal{M}^{\theta_W}(\#^g S^n \times S^n \# W)) \longleftarrow H^*(\Omega^\infty \Psi_{\theta_W})$$

is an isomorphism. [Barratt–Priddy, Quillen, Segal '72 for $2n = 0$, Madsen–Weiss '07 for $2n = 2$, Galatius–R-W '17]

The θ_W can be removed from the LHS, complicating the RHS.

Combined with the stability theorems (for $2n \neq 4$) this calculates the cohomology of $\mathcal{M}(W)$ in a stable range depending on $g(W)$.

Examples

The point of such a theorem is that the right-hand side can be approached by standard methods of algebraic topology, and no longer has anything to do with smooth manifolds.

It is particularly easy to do calculations with $\mathbb{Q}$-coefficients.

Corollary. [Madsen–Weiss, "Mumford's conjecture"]

$$H^*(\mathcal{M}^+(\Sigma_g);\mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]$$

in degrees $* \leq \frac{2g-3}{3}$.

Corollary. For $2n \geq 6$ we have

$$H^*(\mathcal{M}(\#^g S^n \times S^n \setminus int(D^{2n}));\mathbb{Q}) = \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$$

in degrees $* \leq \frac{g-3}{2}$, where $\mathcal{B}$ is the basis of monomials of $\mathbb{Q}[e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}]$ of degree $> 2n$ (where $|e| = 2n$, $|p_i| = 4i$).

Discs

Discs and smoothing theory

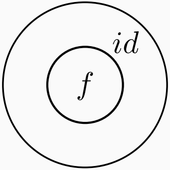
At an opposite extreme we have D^d, the unit disc in $\mathbb{R}^d$.

"Smoothing theory" in the style of Morlet: for $d \neq 4$ have

$$\frac{Homeo(D^d)}{Diff(D^d)} \simeq \Omega^d_0 \left(\frac{Homeo(\mathbb{R}^d)}{O(d)} \right)$$

where $O(d)$ is the orthogonal group.

Alexander trick: For $f : D^d \to D^d$ a homeomorphism fixing ∂D^d, consider



$$f_t(x) = \begin{cases} x & |x| \geq t \\ t \cdot f(x/t) & |x| \leq t. \end{cases}$$

$\Rightarrow Homeo_\partial(D^d) \simeq *$

$\Rightarrow \mathcal{M}(D^d) = BDiff(D^d) \simeq \Omega^d_0 \left(\frac{Homeo(\mathbb{R}^d)}{O(d)} \right)$ for $d \neq 4$

So understanding $\mathcal{M}(D^d)$ is more or less the same as understanding the group of homeomorphisms of $\mathbb{R}^d$. That is clearly important too!

The classical approach

Classical approach to $\mathcal{M}(W) \simeq BDiff(W)$ via

surgery theory and pseudoisotopy theory

Limiting factor comes from dimension d of W (not genus): the "pseudoisotopy stable range" is at least $\min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$ [Igusa '84]

Theorem. [Farrell–Hsiang '78] In this range

$$\pi_*(\mathcal{M}(D^d))_\mathbb{Q} = \begin{cases} 0 & d \text{ even} \\ \mathbb{Q}[4] \oplus \mathbb{Q}[8] \oplus \mathbb{Q}[12] \oplus \cdots & d \text{ odd} \end{cases}$$

The nontrivial classes come from the relation of pseudoisotopy to algebraic K-theory and Borel's calculation

$$K_*(\mathbb{Z})_\mathbb{Q} = \mathbb{Q}[0] \oplus \mathbb{Q}[5] \oplus \mathbb{Q}[9] \oplus \mathbb{Q}[13] \oplus \cdots$$

The theorems of Watanabe and Weiss

More recent work shows that the formula of Farrell and Hsiang cannot be the whole story: there are new phenomena outside of the pseudoisotopy stable range.

Theorem [Watanabe '09 '18] For d even, or d odd and $r > 1$, there is a surjection

$$\pi_{r\cdot(d-3)}(\mathcal{M}(D^d))_{\mathbb{Q}} \longrightarrow \mathcal{A}_r^{(-1)^d}$$

for $\mathcal{A}_r^{\pm}$ certain vector spaces of trivalent graphs. These satisfy

$$\dim(\mathcal{A}_r^-) = 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 9, \ldots$$
$$\dim(\mathcal{A}_r^+) = 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$$

for $r = 1, 2, 3, \ldots$

Theorem [Weiss '22] There are certain maps

$$\pi_{4i-d-1}(\mathcal{M}(D^d))_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

which are surjective for "many" d and $i \geq \frac{d-1}{2}$.

A pattern: even-dimensional discs

Inspired by Weiss' argument, Kupers and I have begun a programme to determine

$$\pi_*(\mathcal{M}(D^{2n}))_{\mathbb{Q}}$$

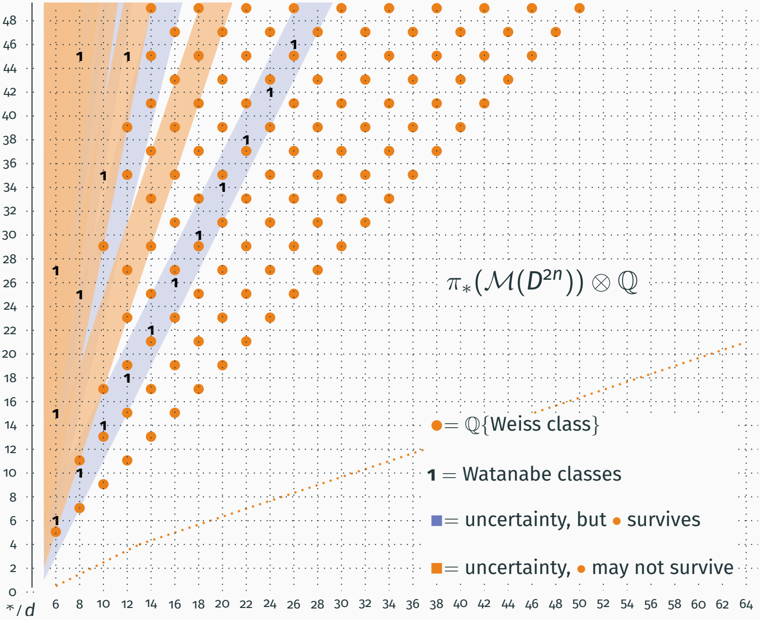
as completely as possible. The first installment is:

Theorem. [Kupers–R-W '20 '21] Let $2n \geq 6$.

(i) If $i < 2n-1$ then $\pi_i(\mathcal{M}(D^{2n}))_{\mathbb{Q}}$ vanishes, and

(ii) if $i \geq 2n-1$ then $\pi_i(\mathcal{M}(D^{2n}))_{\mathbb{Q}}$ is

$$\begin{cases} \mathbb{Q} & \text{if } i \equiv 2n{-}1 \bmod 4 \text{ and } i \notin \bigcup\limits_{r\geq 2} [2r(n{-}2)-1, 2r(n{-}1)+1], \\ 0 & \text{if } i \not\equiv 2n{-}1 \bmod 4 \text{ and } i \notin \bigcup\limits_{r\geq 2} [2r(n{-}2)-1, 2r(n{-}1)+1], \\ ? & \text{otherwise.} \end{cases}$$


$\pi_*(\mathcal{M}(D^{2n})) \otimes \mathbb{Q}$
● = $\mathbb{Q}${Weiss class}
1 = Watanabe classes
■ = uncertainty, but ● survives
■ = uncertainty, ● may not survive
*/ d
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48
6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64

A pattern: odd-dimensional discs

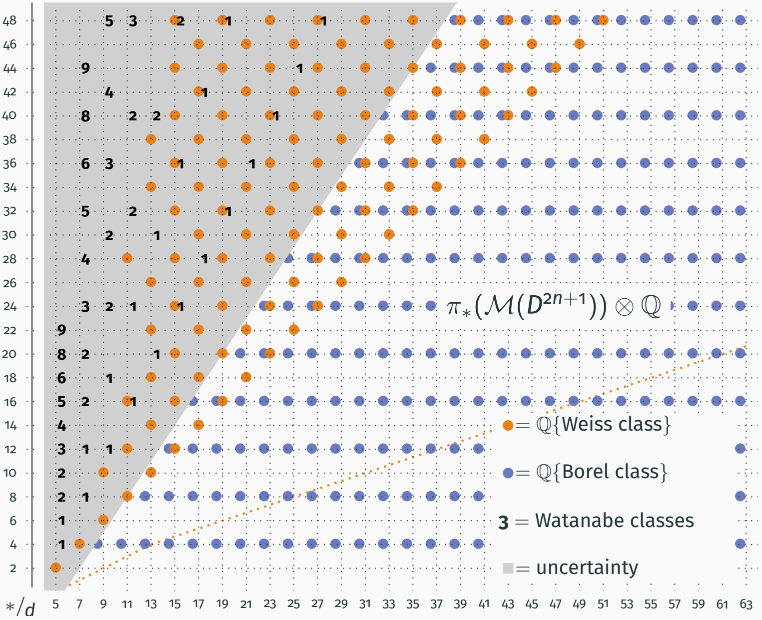
Using different techniques, Krannich and I have investigated

$$\pi_*(\mathcal{M}(D^{2n+1}))_{\mathbb{Q}}$$

outside of the pseudoisotopy stable range.

Theorem. [Krannich–R-W '21] In degrees $i \leq 3n - 8$ we have

$$\pi_i(\mathcal{M}(D^{2n+1}))_{\mathbb{Q}} = K_{i+1}(\mathbb{Z})_{\mathbb{Q}} \oplus \begin{cases} \mathbb{Q} & i \equiv 2n-2 \mod 4, i \geq 2n-2 \\ 0 & \text{else} \end{cases}$$


$\pi_*(\mathcal{M}(D^{2n+1}))\otimes\mathbb{Q}$
= $\mathbb{Q}\{$Weiss class$\}$
= $\mathbb{Q}\{$Borel class$\}$
3 = Watanabe classes
= uncertainty
$*/d$

A conjectural picture

Proposal

The "band" picture suggests that $\pi_*(\mathcal{M}(D^d))_\mathbb{Q}$ is a superposition of various phenomena happening on different "wavelengths"

The phenomena that occur depend only on the parity of d, but the rth phenomenon contributes to degrees around $r \cdot d$

i.e. these phenomena get "spread out" as d increases

There is a mechanism from homotopy theory that could explain this:

Orthogonal Calculus

In fact it is better to think about $BHomeo(\mathbb{R}^d)$ instead of $\mathcal{M}(D^d)$: by smoothing theory these differ in a well understood way.

Then the proposal is to consider all the $BHomeo(\mathbb{R}^d)$ at once, as the functor

$$V \mapsto BHomeo(V) : \left\{\substack{\text{category of finite-dimensional} \\ \text{inner product spaces}}\right\} \longrightarrow \left\{\substack{\text{category of based} \\ \text{topological spaces}}\right\}$$

Orthogonal calculus

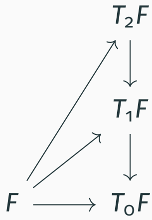
Weiss' orthogonal calculus proposes to consider such functors

$$F : \left\{\begin{smallmatrix}\text{category of finite-dimensional}\\ \text{inner product spaces}\end{smallmatrix}\right\} \longrightarrow \left\{\begin{smallmatrix}\text{category of based}\\ \text{topological spaces}\end{smallmatrix}\right\}$$

as though they were functions, and develop a notion of Taylor expansions for them.

There is a notion of derivative $F^{(1)}(V) := \mathrm{fibre}(F(V) \to F(V \oplus \mathbb{R}))$ of such a functor, and hence of being polynomial of degree $\leq r$.

Any functor F has a best approximation $F \to T_r F$ by a polynomial functor of degree $\leq r$, assembling to a "Taylor tower".



One remarkable thing about this theory is that the homogeneous polynomials, i.e. the fibres of $T_r F \to T_{r-1}F$, have a very particular structure: they are

$$V \longmapsto \Omega^\infty(\Theta F^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$$

for some $O(r)$-spectrum $\Theta F^{(r)}$.

Orthogonal calculus for $V \mapsto BHomeo(V)$

Such a homogeneous functor $F(V) = \Omega^\infty(\Theta F^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$ has precisely the behaviour we have observed

$$\begin{aligned} \pi_*(F(V \oplus \mathbb{R} \oplus \mathbb{R}))_\mathbb{Q} &= \pi_{*-2r}(F(V))_\mathbb{Q} \\ &\neq \pi_{*-r}(F(V \oplus \mathbb{R}))_\mathbb{Q} \text{ in general} \end{aligned}$$

The "band" pattern we have seen would then be explained by

(i) $BHomeo(V) \xrightarrow{\sim} T_\infty BHomeo(V)$ for $\dim(V)$ large enough
(ii) the known structure of $T_0 BHomeo(-)$ and $T_1 BHomeo(-)$
(iii) $\Theta BHomeo^{(r)} /\!\!/ SO(r)$ being a finite spectrum for each $r \geq 2$

The spectra $\Theta BHomeo^{(r)} /\!\!/ SO(r)$ would have to be very rich, with rational homotopy groups at least containing the r-loop part of Kontsevich's (even and odd) commutative graph cohomology, and most probably just being equal to this.

Krannich and I have identified $\Theta BHomeo^{(2)} \simeq_\mathbb{Q} \mathrm{CoInd}_{O(1)}^{O(2)} \mathbb{S}^{-1}$, and this proposal looks good here.

Questions?