# Diffeomorphisms of discs

Oscar Randal-Williams

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# Smoothing theory

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Recording germs of smooth structure near each point gives a map

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For $d \neq 4$ this map is a homotopy equivalence.

$Homeo_\partial(M)$ acts on $\mathcal{S}m(M)$, giving

$$\mathcal{S}m(M) \cong \coprod_{[W]} Homeo_\partial(W)/Diff_\partial(W)$$

Similarly, $\mathcal{S}m(\mathbb{R}^d) \cong Homeo(\mathbb{R}^d)/Diff(\mathbb{R}^d)$

# A consequence of smoothing theory

Write $\mathit{Top}(d) := \mathit{Homeo}(\mathbb{R}^d)$. By linearising have $\mathit{Diff}(\mathbb{R}^d) \simeq O(d)$, so

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Applied to $D^d$, $d \neq 4$, smoothing theory gives a map

$$Homeo_\partial(D^d)/Diff_\partial(D^d) \longrightarrow \Gamma_\partial(\mathcal{S}m(TD^d) \to D^d) = map_\partial(D^d, Top(d)/O(d))$$

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The Alexander trick $Homeo_\partial(D^d) \simeq *$ implies

$$BDiff_\partial(D^d) \simeq \Omega^d_0 Top(d)/O(d) \qquad \text{(Morlet)}$$

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$O(d)$ is “well understood” so $Diff_\partial(D^d)$ and $Top(d)$ are equidifficult.

But $Diff_\partial(D^d)$ is more approachable: can *use* smoothness.

# What do we know?

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**Theorem**. [Farrell–Hsiang '78]

$$\pi_*(BDiff_\partial(D^d)) \otimes \mathbb{Q} = \begin{cases} 0 & d \text{ even} \\ \mathbb{Q}[4] \oplus \mathbb{Q}[8] \oplus \mathbb{Q}[12] \oplus \cdots & d \text{ odd} \end{cases}$$

in the pseudoisotopy stable range for $d$ (so certainly for $* \lesssim \frac{d}{3}$).

# The theorems of Watanabe

**Theorem.** [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

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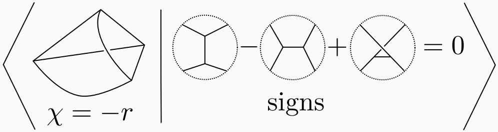

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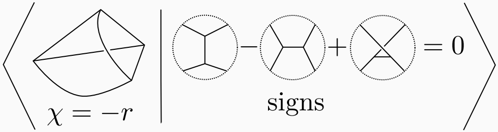
$\mathcal{A}_r^{odd} =$ 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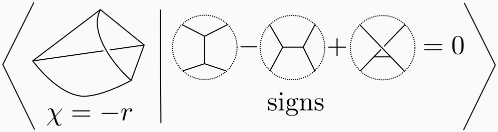


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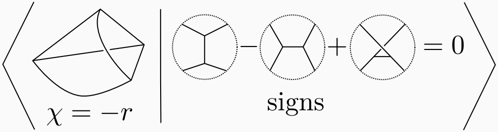


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where $dim(\mathcal{A}_r^{even}) = 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$ (so $\pi_2(BDiff_\partial(D^4)) \neq 0$)

# The theorem of Weiss

Closely related to the classical story is the fact that the stable map

$$O = \operatorname*{colim}_{d \to \infty} O(d) \longrightarrow Top = \operatorname*{colim}_{d \to \infty} Top(d)$$

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**Theorem.** [Weiss '15]
For many $n$ and $i \geq 0$ there are classes $w_{n,i} \in \pi_{4(n+i)}(BTop(2n))$ which pair nontrivially with $p_{n+i}$ (i.e. (!) does not hold on $BTop(2n)$).

$\Rightarrow \pi_{2n-1+4i}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \neq 0$ for such $n$ and $i$.

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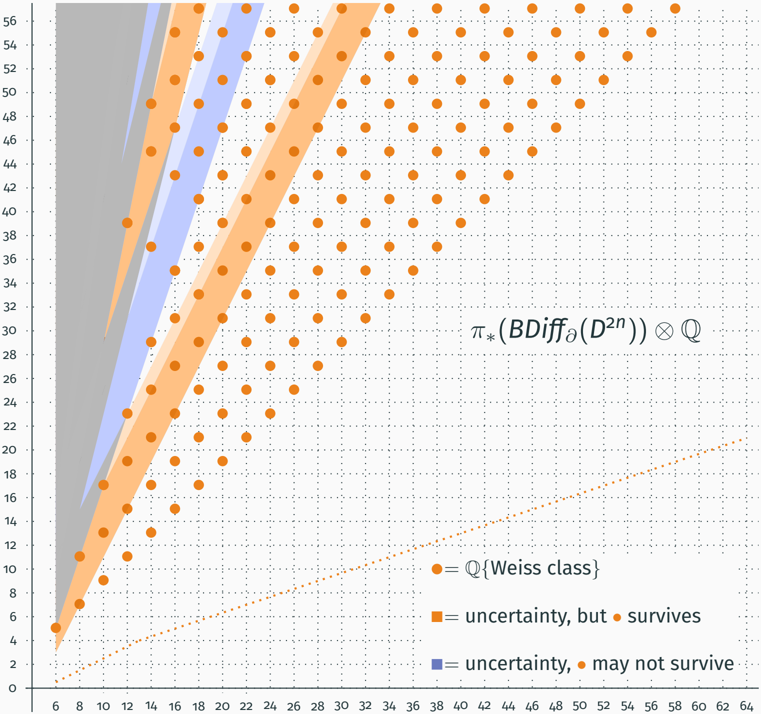
3. understand something about the structure of these bands.



$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$
= $\mathbb{Q}${Weiss class}
= uncertainty, but survives
= uncertainty, may not survive

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**Theorem.** [Kupers–R-W]
Let $2n \geq 6$.

(i) If $d < 2n - 1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ vanishes, and

(ii) if $d \geq 2n - 1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ is

$$\begin{cases} \mathbb{Q} & \text{if } d \equiv 2n{-}1 \text{ mod } 4 \text{ and } d \notin \bigcup_{r\geq 2} [2r(n{-}2) - 1, 2rn - 1], \\ 0 & \text{if } d \not\equiv 2n{-}1 \text{ mod } 4 \text{ and } d \notin \bigcup_{r\geq 2} [2r(n{-}2) - 1, 2rn - 1], \\ ? & \text{otherwise.} \end{cases}$$

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Using the fibre sequence $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

**Reformulation (slightly stronger).**
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

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Reflecting $D^{2n}$ or $\mathbb{R}^{2n}$ induces compatible involutions on

$$\Omega_0^{2n+1} \frac{Top}{Top(2n)} \longrightarrow BDiff_\partial(D^{2n}) \simeq \Omega_0^{2n} \frac{Top(2n)}{O(2n)} \longrightarrow \Omega_0^{2n} \frac{Top}{O(2n)}.$$

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We show this acts as $-1$ on

$$\pi_*(\Omega_0^{2n} \frac{Top}{O(2n)}) \otimes \mathbb{Q} = \mathbb{Q}[2n-1] \oplus \mathbb{Q}[2n+3] \oplus \mathbb{Q}[2n+7] \oplus \cdots$$

and acts on $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ as $(-1)^r$ in the $r$th band.

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The orange/blue colours in the chart are the $+1/-1$ eigenspaces.

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By analogy with Watanabe's theorem for $D^4$ one expects

$$dim\, \pi_{4n-6}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \geq 1$$

which is compatible with the above.

# Remarks on the proof

# Philosophy

Many results in this flavour of geometric topology are *relative*: they describe the difference between

1. topological/smooth manifolds (smoothing)
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Weiss suggested a new kind of relativisation:

for $M$ with $\partial M = S^{d-1}$ and $\frac{1}{2}\partial M := D^{d-1} \subset S^{d-1}$ he showed that

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(The "codimension" of such embeddings can be $\geq 3$.)

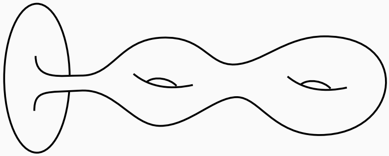
**Strategy:** find a manifold $M$ for which one can understand $Emb^{\cong}_{1/2\partial}(M)$ and $Diff_\partial(M)$, then deduce things about $Diff_\partial(D^d)$.

# The manifold $W_{g,1}$

A good choice is

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$

especially for “arbitrarily large” $g$.



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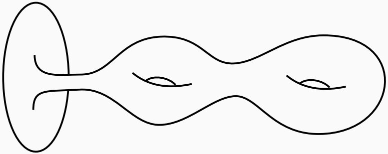

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**Theorem.** [Madsen–Weiss '07 $2n = 2$, Galatius–R-W '14 $2n \geq 4$]

$$\lim_{g \to \infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$$

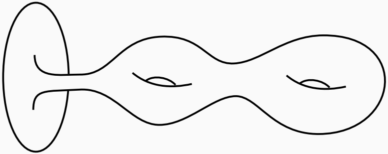
Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

# The manifold $W_{g,1}$

A good choice is

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$

especially for “arbitrarily large” $g$.



**Theorem.** [Madsen–Weiss ’07 $2n = 2$, Galatius–R-W ’14 $2n \geq 4$]

$$\lim_{g\to\infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$$

Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

**Theorem.** [Berglund–Madsen ’20 $2n \geq 6$]

$$\lim_{g\to\infty} H^*(\widetilde{BDiff}_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\tilde{\kappa}_c^\xi \mid (c, \xi) \in \mathcal{B}']$$

$$\lim_{g\to\infty} H^*(BhAut_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\tilde{\kappa}_c^\xi \mid (c, \xi) \in \mathcal{B}'']$$

Here $\mathcal{B}'$ and $\mathcal{B}''$ are much more complicated than $\mathcal{B}$, and we will probably never be able to enumerate them completely.

# Difficulties I

Embedding calculus describes $Emb^{\cong}_{1/2\partial}(W_{g,1})$ as the limit of a tower

$$\begin{array}{ccccc} T_1Emb^{\cong}_{1/2\partial}(W_{g,1}) & \longleftarrow & T_2Emb^{\cong}_{1/2\partial}(W_{g,1}) & \longleftarrow & T_3Emb^{\cong}_{1/2\partial}(W_{g,1}) \cdots \\ & & \big\uparrow & & \big\uparrow \\ & & L_2Emb^{\cong}_{1/2\partial}(W_{g,1}) & & L_3Emb^{\cong}_{1/2\partial}(W_{g,1}) \end{array}$$

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The term $T_1Emb^{\cong}_{1/2\partial}(W_{g,1})$ is close to being the space of homotopy self-equivalences of $W_{g,1}$ relative to half the boundary; if we instead use *framed* self-embeddings then it is:

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By rational homotopy theory, for $L := Lie(s^{-1}H_n(W_{g,1};\mathbb{Q}))$ have

$$\pi_{*>0}(hAut_{1/2\partial}(W_{g,1})) \otimes \mathbb{Q} = Der^{+}(L, L) = Hom_{\mathbb{Q}}(s^{-1}H_n(W_{g,1};\mathbb{Q}), L),$$

supported in degrees which are multiples of $n-1$.

# Difficulties I

The higher layers are described as spaces of sections

$$L_k Emb^{\cong}_{1/2\partial}(W_{g,1}) \simeq \Gamma_\partial \left( \begin{array}{ccc} Z_k & \longleftarrow & tohofib_{I \subseteq [k]} Emb(I, W_{g,1}) \\ \downarrow & & \\ Conf_k(W_{g,1}) & & \end{array} \right)$$

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The homotopy groups of such a space can be computed by a twisted form of the Federer spectral sequence. Rationally express this as

$$E^2_{p,q} \otimes \mathbb{Q} = [H^p(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q}) \otimes \pi_q(tohofib_{I\subseteq[k]} Emb(I, W_{g,1}))]^{\mathfrak{S}_k}$$
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The main issue is to determine/estimate the characters of

$$H^p(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q}) \quad \text{and} \quad \pi_q(Emb([k], W_{g,1})) \otimes \mathbb{Q}$$

as representations of $\mathfrak{S}_k \times \pi_0(\mathrm{Diff}_\partial(W_{g,1}))$.

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The character of $H^p(W^k_{g,1}, \Delta_{1/2\partial}; \mathbb{Q})$ can be determined easily using a theorem of Petersen '20.

The character of $\pi_q(Emb([k], W_{g,1})) \otimes \mathbb{Q}$ is much more complicated, but we are able to get a closed expression for it by relating it to an extended form of the Drinfel'd–Kohno Lie algebra, using Koszul duality, recognising the Koszul dual, and working with this.

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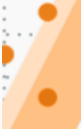
Nonetheless this lets us prove that $\pi_*(Emb^{\cong, fr}_{1/2\partial}(W_{g,1})) \otimes \mathbb{Q}$ is supported in degrees $* \in \cup_{r \geq 1}[r(n-2)-1, r(n-1)]$. This is the darkly shaded region in the chart.



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$\pi_1(BDiff_\partial(W_{g,1}))\sim Sp_{2g}(\mathbb{Z})$ ($n$ odd) or $O_{g,g}(\mathbb{Z})$ ($n$ even)
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Can pass to the Torelli subgroup

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In two companion papers we prove that the space $BTor_\partial(W_{g,1})$ is nilpotent, and determine $H^*(BTor_\partial(W_{g,1});\mathbb{Q})$ as $g\to\infty$.

A. Kupers, O. R-W, *On the cohomology of Torelli groups*
Forum of Mathematics, Pi, 8 (2020)
A. Kupers, O. R-W, *The cohomology of Torelli groups is algebraic*
Forum of Mathematics, Sigma, to appear

# Difficulties II

Adapting this to the framed case, we produce a fibration

$$X_1(g) \longrightarrow BTor^{fr}_{\partial}(W_{g,1}) \longrightarrow X_0$$

with $H^*(X_0; \mathbb{Q}) = \Lambda_{\mathbb{Q}}[\bar{\sigma}_{4j-2n-1} \,|\, j > n/2]$.

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We show that in a stable range, $H^*(X_1(g); \mathbb{Q})$ is generated by classes

$$\kappa(v_1 \otimes \cdots \otimes v_r) \in H^{(r-2)n}(X_1(g); \mathbb{Q}) \qquad r \geq 3, \quad v_i \in H^n(W_{g,1}; \mathbb{Q})$$

subject only to the relations (where $\{a_i\}$ and $\{a_i^\#\}$ are dual bases)

(i) linearity in each $v_i$,
(ii) $\kappa(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa(v_1 \otimes v_2 \otimes \cdots \otimes v_r)$,
(iii) $\sum_i \kappa(v \otimes a_i) \cdot \kappa(a_i^\# \otimes w) = \kappa(v \otimes w)$, for any tensors $v$ and $w$,
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The unstable Adams spectral sequence then shows

$$\pi_*(BTor_\partial^{fr}(W_{g,1}))\otimes\mathbb{Q} = \left(\bigoplus_{j>n/2} \mathbb{Q}[4j-2n-1]\right) \text{“}\oplus\text{”} \begin{pmatrix}\text{something supported in} \\ *\in\bigcup_{r\geq 0}[r(n-1)+1, rn-2]\end{pmatrix}$$

The second piece is the lightly shaded region in the chart.

# Optimism

# Divergent embedding calculus

Can apply embedding calculus to diffeomorphisms, considered as codimension 0 embeddings. It need not converge and in fact does not converge: by work of Fresse, Turchin, and Willwacher '17 it predicts (modulo a subtlety) that $\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ should be

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so misses the Weiss classes and starts with some spurious classes. But apart from this it has classes supported in our bands, and here is given precisely by Kontsevich's graph complex $\mathrm{GC}^2_{2n}$.

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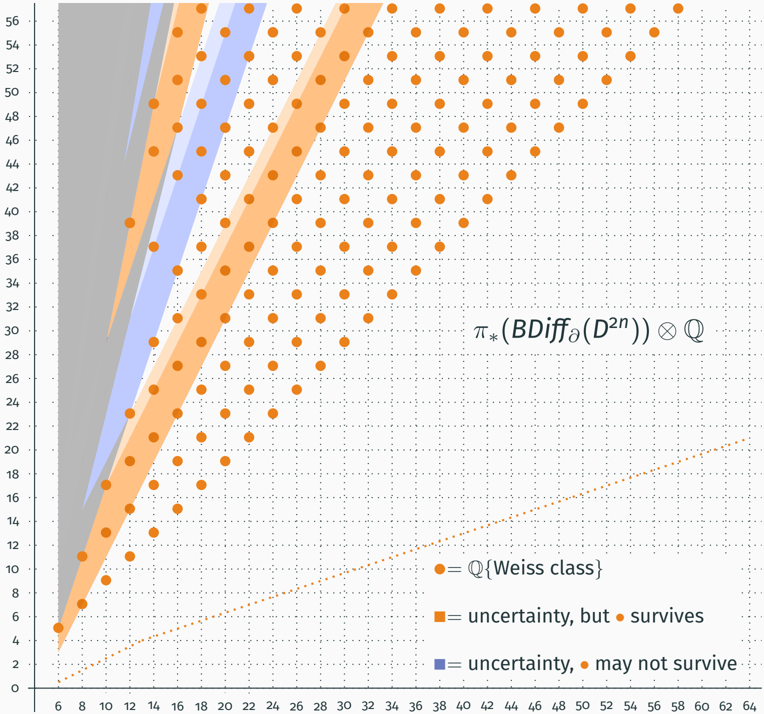
**Evidence.** [Prigge '20]
The family signature theorem does not hold on $BT_2 Diff_\partial(M)$.

# Questions?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