E_∞-algebras and general linear groups

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Premise

All based on joint work with S. Galatius and A. Kupers,
E_∞-cells and general linear groups of infinite fields

We want to study the homology of $GL_n(A)$ for various rings A, especially the behaviour with respect to varying n.

To do so, consider the totality

$$\mathbf{R}^+ = \coprod_{n \geq 0} BGL_n(A),$$

which is a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces.

We have tried to understand cellular E_∞-algebra structures on $\mathbf{R}^+$, and in doing so have been led to many results which can be stated without reference to E_∞-algebras.

I will first explain some of these results, and later give an idea of how they all fit together.

The Steinberg module

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The unique homology group

$$St(V) := \widetilde{H}_{\dim(V)-2}(T(V); \mathbb{Z})$$

is the "Steinberg module". As $GL(V)$ acts on $T(V)$, it acts on $St(V)$.

Bilinear forms on the Steinberg module

As $T(V)$ is a $(\dim(V) - 2)$-dimensional simplicial complex, $St(V)$ is a submodule of $\widetilde{C}_{\dim(V)-2}(T(V); \mathbb{Z})$.

This has a basis of complete flags in V: give it a bilinear form by declaring the complete flags to be orthonormal.

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This restricts to a positive-definite symmetric bilinear form

$$\langle -, - \rangle : St(V) \otimes St(V) \longrightarrow \mathbb{Z},$$

e.g. if a is an “apartment” then $\langle a, a \rangle = \dim(V)!$.

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Corollary (Galatius–Kupers–R-W)
For any connected commutative ring $\mathbb{k}$*, the* $\mathbb{k}[GL(V)]$*-module* $\mathbb{k} \otimes_{\mathbb{Z}} St(V)$ *is indecomposable.*

Bilinear forms on the Steinberg module

There are natural multiplication maps $St(V) \otimes St(W) \to St(V \oplus W)$, which give

$$\mathbb{Z}\{1\} \oplus \bigoplus_{n \geq 1} [St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})}$$

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Theorem (Galatius–Kupers–R-W)
The isomorphisms $[St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})} \xrightarrow{\sim} \mathbb{Z}$ assemble to a ring isomorphism

$$\mathbb{Z}\{1\} \oplus \bigoplus_{n \geq 1} [St(\mathbb{F}^n) \otimes St(\mathbb{F}^n)]_{GL_n(\mathbb{F})} \cong \Gamma_{\mathbb{Z}}[x]$$

to a divided power algebra i.e. $\langle 1, \frac{x^1}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots \rangle_{\mathbb{Z}} \subset \mathbb{Q}[x]$.

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The proofs of these results use a presentation of $St(\mathbb{F}^n)$ due to Lee–Szczarba, and elementary but complicated manipulations of matrices.

Rognes’ connectivity conjecture

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Let A be a connected commutative ring for which f.g. projective modules are free. (e.g. a field or local ring)

Rogens has defined a "rank filtration"

$$* \subset F_0\mathbf{K}(A) \subset F_1\mathbf{K}(A) \subset F_2\mathbf{K}(A) \subset \cdots \subset \mathbf{K}(A)$$

of the algebraic K-theory spectrum of A, and has identified the filtration quotients

$$\frac{F_n\mathbf{K}(A)}{F_{n-1}\mathbf{K}(A)} \simeq \mathbf{D}(A^n)_{hGL_n(A)}$$

as the homotopy orbits of certain $GL_n(A)$-spectra $\mathbf{D}(A^n)$, the nth *stable building* of A.

(Idea: the Tits building is the first space in this spectrum; the kth space is made from k-dimensional flags of submodules of A^n.)

Based on calculations for $n \leq 3$, Rognes conjectured that for A local or Euclidean the spectrum $\mathbf{D}(A^n)$ is $(2n-3)$-connected.

Rognes' connectivity conjecture

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Theorem (Galatius–Kupers–R-W)

(i) *If A is a connected semi-local ring with all residue fields infinite, then $\mathbf{D}(A^n)_{hGL_n(A)}$ is $(2n-3)$-connected.*

(ii) *If A is an infinite field then in addition*

$$H_{2n-2}(\mathbf{D}(A^n)_{hGL_n(A)}) = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ \mathbb{Z}/p & \text{if } n = p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

Rognes had also conjectured that $H_0(GL_n(A); H_{2n-2}(\mathbf{D}(A^n)))$ is torsion for $n > 1$, which aligns with (ii).

Homological stability

What is homological stability?

Have stabilisation maps

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : GL_{n-1}(A) \longrightarrow GL_n(A)$$

and *homological stability* hopes these are homology isomorphisms in a range of degrees going to ∞ with n.

Equivalently, it hopes that

$$H_d(GL_n(A), GL_{n-1}(A)) = 0 \text{ for all } d \leq f(n)$$

for some divergent function f.

One can ask this question for homology with $\mathbb{k}$-coefficients: the function f may then depend on $\mathbb{k}$.

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Stability with $\mathbb{Z}$-coefficients is known when A has finite "stable rank", by work of Maazen and van der Kallen: then $f(n) = \frac{n - sr(A)}{2}$ will do.

The Nesterenko–Suslin theorem

Sometimes one has homological stability in a range of degrees much larger than the slope $\frac{1}{2}$ range of Maazen and van der Kallen.

Nesterenko–Suslin: If A is a local ring with infinite residue field then

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Z}) = 0 \text{ for } d < n,$$

and $H_n(GL_n(A), GL_{n-1}(A); \mathbb{Z}) \cong K_n^M(A)$, nth Milnor K-theory.

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Recall: Milnor K-theory $K_*^M(A)$ is the graded ring generated by $K_1^M(A) = A^\times$ and subject to the relations $a \cdot b = 0 \in K_2^M(A)$ whenever $a, b \in A^\times$ satisfy $a + b = 1$. (A calculation shows it is graded commutative.)

A degree above the Nesterenko–Suslin theorem

We study these relative homology groups one degree further up (and rationally). We first show that

$$\bigoplus_{n\geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

can be made into a $K_*^M(A) \otimes \mathbb{Q}$-module, then analyse how it may be generated efficiently.

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Theorem (Galatius–Kupers–R-W)
If A is a connected semi-local ring with all residue fields infinite, then there is a map of graded $\mathbb{Q}$-vector spaces

$$\mathrm{Harr}_3(K^M_*(A) \otimes \mathbb{Q}) \longrightarrow \mathbb{Q} \otimes_{K^M_*(A) \otimes \mathbb{Q}} \bigoplus_{n \geq 1} H_{n+1}(GL_n(A), GL_{n-1}(A); \mathbb{Q})$$

which is an isomorphism in gradings ≥ 5.

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which is an isomorphism in gradings ≥ 5.

Here Harr = Harrison homology = André–Quillen homology.
Third Harrison homology measures "relations between relations" in a presentation of the quadratic algebra $K^M_*(A) \otimes \mathbb{Q}$.

Improved homological stability

Under further assumptions on A, our methods (which I have not yet told you) instead give improved homological stability results:

Theorem (Galatius–Kupers–R-W)

(i) *If A is a connected semi-local ring with all residue fields infinite and such that $K_2(A) \otimes \mathbb{Q} = 0$ (e.g. $\bar{\mathbb{F}}_q$, $\mathbb{F}_q(t)$, number field, $\bar{\mathbb{Q}}$) then*

$$H_d(GL_n(A), GL_{n-1}(A); \mathbb{Q}) = 0 \text{ for } d < \tfrac{4n-1}{3}.$$

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(ii) *If A is a connected semi-local ring with all residue fields infinite and p is a prime number such that $A^\times \otimes \mathbb{Z}/p = 0$ then*

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(iii) *If $\mathbb{F}$ is an algebraically closed field then, for all primes p,*

$$H_d(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0 \text{ for } d < \tfrac{5n}{3}.$$

Resolving some conjectures

The last part implies that if $\mathbb{F}$ is an algebraically closed field then

$$H_{n+1}(GL_n(\mathbb{F}), GL_{n-1}(\mathbb{F}); \mathbb{Z}/p) = 0$$

for all $n > 1$ and all primes p.

This resolves a conjecture of Mirzaii on certain “higher pre-Bloch groups” $\mathfrak{p}_n(\mathbb{F})$, and a conjecture of Yagunov on a different notion of “higher pre-Bloch groups” $\wp_n(\mathbb{F})$ and $\wp_n(\mathbb{F})_{cl}$.

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In a different direction, we can complete an approach of Mirzaii to proving Suslin's "injectivity conjecture":

Theorem (Galatius–Kupers–R-W)
If $\mathbb{F}$ is an infinite field and $\mathbb{k}$ is a field in which $(n-1)!$ is invertible then the stabilisation map

$$H_n(GL_{n-1}(\mathbb{F}); \mathbb{k}) \longrightarrow H_n(GL_n(\mathbb{F}); \mathbb{k})$$

is injective.

Homology of Steinberg modules

Finally, and returning to the beginning, we prove a vanishing theorem for the homology of the Steinberg module.

Theorem (Galatius–Kupers–R-W)
If A is a connected semi-local ring with infinite residue fields, then

$$H_d(GL_n(A); St(A^n)) = 0$$

for $d < \frac{1}{2}(n-1)$.

Analogous results in the case of fields have been obtained by Ash–Putman–Sam and Miller–Nagpal–Patzt.

What do these things have to do with each other?

Reformulation

These results arose in our analysis of

$$\mathbf{R}^+ = \bigsqcup_{n \geq 0} BGL_n(A)$$

as a unital E_∞-algebra in the category of $\mathbb{N}$-graded spaces.

Rognes' conjecture and our description of bilinear forms on the Steinberg module essentially corresponds to computing the "E_2-homology" of $\mathbf{R}^+$ in a range of degrees.

This determines the "E_∞-homology" of $\mathbf{R}^+$ in this range of degrees, implying that there is a cell structure on $\mathbf{R}^+$ in the category of E_∞-algebras with highly constrained cells.

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The applications to homological stability are calculations using this constrained cell structure and the description

$$H_d(GL_n(A), GL_{n-1}(A)) = H_{n,d}(\mathbf{R}^+/\sigma)$$

for $\sigma \in H_0(BGL_1(A))$. I will not describe these calculations today.

Homotopy theory of E_k-algebras

Graded objects

Let C denote sSet, sSet_*, Sp, or (because we are eventually interested in taking $\mathbb{k}$-homology) $\mathrm{sMod}_{\mathbb{k}}$.

Write $\otimes$ for the cartesian, smash, or tensor product.

We will consider $\mathbb{N}$-graded objects in C, meaning $\mathrm{C}^{\mathbb{N}} := \mathrm{Fun}(\mathbb{N}, \mathrm{C})$. This is given the Day convolution monoidal structure:

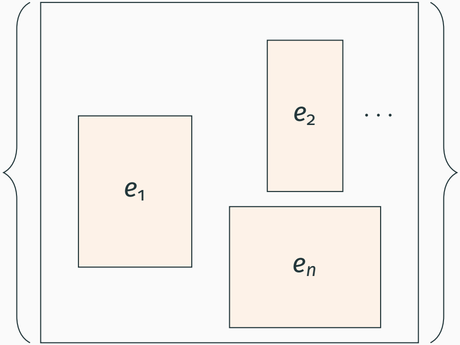
$$(X \otimes Y)(n) = \bigsqcup_{a+b=n} X(a) \otimes Y(b).$$

Define bigraded homology groups as $H_{n,d}(X; \mathbb{k}) := H_d(X(n); \mathbb{k})$.

E_k-algebras

Let $\mathcal{C}_k$ denote the non-unital ($\mathcal{C}_k(0) = \varnothing$) little k-cubes operad.

$\mathcal{C}_2(n) =$



The categories $\mathsf{C}^{\mathbb{N}}$ are all tensored over Top: can make sense of the monad

$$E_k(X) := \coprod_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n}$$

and so of E_k-algebras $\mathbf{X}$ in $\mathsf{C}^{\mathbb{N}}$. Call the category of these $\mathrm{Alg}_{E_k}(\mathsf{C}^{\mathbb{N}})$.

E_k-indecomposables

For $\mathbf{X} \in \mathrm{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}})$ define the *E_k-indecomposables* of $\mathbf{X}$ by

$$E_k(X) = \bigsqcup_{n \geq 1} \mathcal{C}_k(n) \odot_{\mathfrak{S}_n} X^{\otimes n} \underset{c}{\overset{\mu_X}{\rightrightarrows}} X \longrightarrow Q^{E_k}(\mathbf{X})$$

where c collapses all factors with $n > 1$ to the basepoint, and applies the augmentation $\varepsilon : \mathcal{C}_k(1)_+ \to S^0$.

Q^{E_k} is left adjoint to the inclusion $\mathsf{C}_*^{\mathbb{N}} \to \mathrm{Alg}_{E_k}(\mathsf{C}_*^{\mathbb{N}})$ by imposing the trivial E_k-action.

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This construction is not homotopy invariant, so we should instead evaluate the derived functor

$$Q^{E_k}_{\mathbb{L}}(\mathbf{X}) := Q^{E_k}(\text{cofibrant replacement of } \mathbf{X}),$$

a.k.a. topological Quillen homology (for the operad $\mathcal{C}_k$).

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Write $H^{E_k}_{n,d}(\mathbf{X}) = H_{n,d}(Q^{E_k}_{\mathbb{L}}(\mathbf{X}))$, the "$E_k$-homology".

Computing derived E_k-indecomposables

$Q^{E_k}_{\mathbb{L}}(\mathbf{X})$ may also be computed by a k-fold bar construction.

Instances of this have previously been given by Getzler–Jones, Basterra–Mandell, Fresse, Francis.

Specifically, if $\mathbf{X}$ is an E_k-algebra with unitalisation $\mathbf{X}^+$, then there is an equivalence

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Considering the k-fold bar construction as the bar construction of the $(k-1)$-fold bar construction gives a bar spectral sequence

$$E^2_{n,p,q} = Tor_p^{H_{*,*}(B^{E_{k-1}}(\mathbf{X}^+);\mathbb{k})}(\mathbb{k}, \mathbb{k})_{n,q} \Rightarrow H_{n,p+q}(B^{E_k}(\mathbf{X}^+);\mathbb{k}).$$

This allows one, in principle, to calculate the E_k-homology by taking iterated bar constructions.

The general linear groups

The general linear group E_∞-algebra

Let A be a connected commutative ring for which f.g. projective modules are free.

The symmetric monoidal category P_A of f.g. projective A-modules and their isomorphisms has classifying space

$$\mathbf{R}^+ = B\mathsf{P}_A \simeq \coprod_{n \geq 0} BGL_n(A)$$

and is equipped with an action of an E_∞-operad. We consider this as $\mathbb{N}$-graded via the rank functor $r : \mathsf{P}_A \to \mathbb{N}$.

The general linear group E_∞-algebra

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We indeed have $\mathbf{R}^+(n) \simeq \mathrm{colim}_{r/n}\, \mathbf{T} = \mathbf{T}(A^n)/GL_n(A) \simeq BGL_n(A)$.

The E_k-splitting complexes

The advantage of the second description is that many constructions commute with Kan extension: we can instead compute them for the simple object $\mathbf{T} \xrightarrow{\sim} \mathbf{t}$ (at the expense of working in the complicated category $\mathrm{sSet}^{\mathsf{P}_A}$).

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In particular, $B^{E_k}(\mathbf{T})$ has the following description: evaluated at a projective module M it is the k-fold simplicial pointed set $\widetilde{D}^k(M)$ with $(p_1, p_2, \ldots, p_k)$-simplices given by

$$\frac{\{M_{i_1, i_2, \ldots, i_k} \leq M \text{ for } 1 \leq i_j \leq p_j\}}{\{\text{those for which } \bigoplus M_{i_1, i_2, \ldots, i_k} \to M \text{ is not an iso}\}}$$

and face maps given by direct sum of submodules and degeneracies given by inserting trivial modules.

Thus we have $(\Sigma^k Q^{E_k}_{\mathbb{L}}(\mathbf{R}))(n) \simeq \widetilde{D}^k(A^n)_{hGL_n(A)}$ for $n > 0$.

The k-fold Tits building

Rognes defines a k-fold analogue of the Tits building for M as the k-fold simplicial pointed set $D^k(M)$ with $(p_1, p_2, \ldots, p_k)$-simplices given by

$$\frac{\{\text{lattices } \varphi : [p_1] \times \cdots \times [p_k] \to \mathrm{Sub}(M)\}}{\{\text{non-full lattices}\}}$$

where a “lattice” is a functor to the poset of direct summands of M such that

$$\operatorname*{colim}_{[a_1 \le b_1] \times \ldots \times [a_k \le b_k] \setminus \{b\}} \varphi \longrightarrow \varphi(b)$$

is an isomorphism onto a direct summand, and a lattice is “full” if $\varphi(a_1, \ldots, a_k) = 0$ whenever some $a_i = 0$, and $\varphi(p_1, ..., p_k) = M$.

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When $k = 1$ and A is a field we have $D^1(A^n) \simeq \Sigma^2 T(A^n)$, the double suspension of the Tits building. By the Solomon–Tits theorem, this is a wedge of n-spheres.

The key theorem

Theorem (Galatius–Kupers–R-W)
If A is a field then the natural map $D^2(M) \to D^1(M) \wedge D^1(M)$ is an isomorphism, and so $D^2(A^n)$ is a wedge of $2n$-spheres.

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The proof in the first case is completely elementary. It also gives

$$\widetilde{H}_{2n}(D^2(A^n)) = St(A^n) \otimes St(A^n),$$

which explains our interest in computing the coinvariants of this.

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The proof in the second case is far more complicated, involving the contractibility of a complex of "submodules in general position".

Rings with many units

There are maps $\widetilde{D}^k(M) \to D^k(M)$ given by sending a k-fold splitting to the associated k-fold flag. These are never isomorphisms, but we have the following:

Theorem (Galatius–Kupers–R-W)
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$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

induce isomorphisms on group homology (which is what we need).

Proof of Rognes' conjecture

$D^2(A^n)$ is $(2n-1)$-connected by the Key Theorem, so $D^2(A^n)_{hGL_n(A)}$ is also $(2n-1)$-connected, and this is the same as $\widetilde{D}^2(A^n)_{hGL_n(A)}$.

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The first results I mentioned say that these are $\mathbb{Z}$ and combine to form a divided power algebra $\Gamma_{\mathbb{Z}}[x]$. The bar spectral sequence shows that

$$\widetilde{H}_{2n+1}(\widetilde{D}^3(A^n)_{hGL_n(A)}) = Tor_1^{\Gamma_{\mathbb{Z}}[x]}(\mathbb{Z},\mathbb{Z})_n = \begin{cases} \mathbb{Z} & \text{if } n=1, \\ \mathbb{Z}/p & \text{if } n=p^k \text{ with } p \text{ prime}, \\ 0 & \text{otherwise.} \end{cases}$$

E_∞-homology

We have $H^{E_k}_{n,d}(\mathbf{R}) = H_{n,d+k}(\widetilde{D}^k(A^n)_{hGL_n(A)})$ and so taking colimits

$$H^{E_\infty}_{n,d}(\mathbf{R}) = H^{spec}_{n,d}(\mathbf{D}(A^n)_{hGL_n(A)}),$$

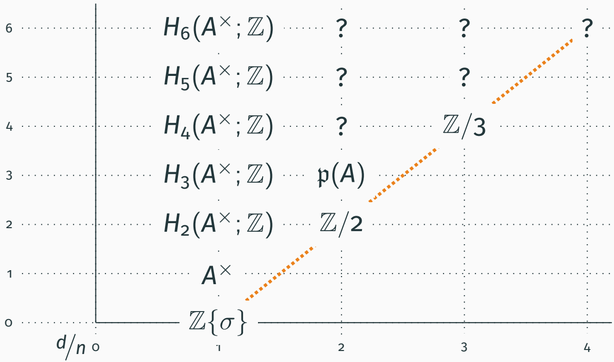
which we have just shown vanishes for $d < 2(n-1)$.

Furthermore, if A is a field then we have computed $H^{E_\infty}_{n,2(n-1)}(\mathbf{R})$.

In fact, we also show the latter calculation is valid for connected semi-local rings with infinite residue fields as long as $n \leq 3$; we conjecture that for such rings it is valid for all n.

E_∞-homology

Combining the vanishing line for E_∞-homology with calculations of Suslin for $GL_2(A)$, we obtain the following chart for E_∞-homology:



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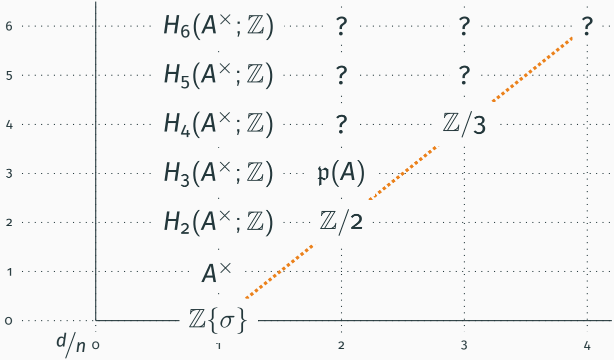

$\mathfrak{p}(A)$ = "pre-Bloch group": generated by $[x] \in A^\times \setminus \{1\}$ subject to

$$[x] - [y] + \left[\frac{y}{x}\right] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0$$

whenever x, y, $1-x$, $1-y$, and $x - y \in A^\times$.

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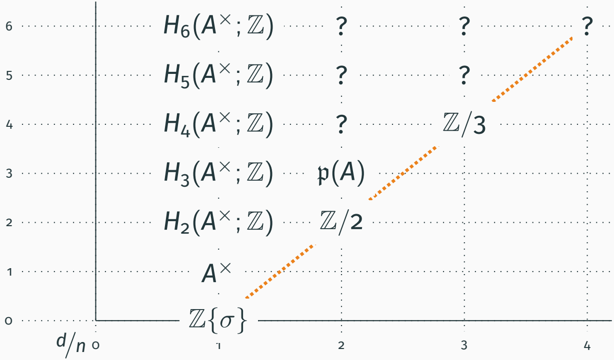

The homological stability theorems are proved by constructing a minimal cellular E_∞-algebra model for $\mathbf{R}^+$ compatible with this chart, and studying its consequences for $\mathbf{R}^+/\sigma$.

Literature

Based on work with S. Galatius and A. Kupers:

E_∞-cells and general linear groups of infinite fields.
arXiv:2005.05620.

Cellular E_k-algebras.
arXiv:1805.07184.

For further applications of these ideas see also:

E_2-cells and mapping class groups.
Publ. Math. Inst. Hautes Études Sci. 130 (2019), 1–61.

E_∞-cells and general linear groups of finite fields.
arXiv:1810.11931.