Diffeomorphisms of discs. I

Oscar Randal-Williams

European Research Council
Established by the European Commission

LEVERHULME
TRUST

Smoothing theory

M a topological d-manifold, maybe with smooth boundary ∂M

$$\mathcal{S}m(M) = \{ \text{ space of smooth structures on } M\text{, fixed near } \partial M \}$$

(“space” interpreted liberally).

Smoothing theory

M a topological d-manifold, maybe with smooth boundary ∂M

$$\mathcal{S}m(M) = \{ \text{ space of smooth structures on } M\text{, fixed near } \partial M \}$$

("space" interpreted liberally).

Recording germs of smooth structure near each point gives a map

$$\mathcal{S}m(M) \longrightarrow \Gamma_\partial(\mathcal{S}m(TM) \to M)$$

(the space of sections of the bundle with fibre $\mathcal{S}m(T_m M) \cong \mathcal{S}m(\mathbb{R}^d)$)

Smoothing theory

M a topological d-manifold, maybe with smooth boundary ∂M

$$\mathcal{S}m(M) = \{ \text{ space of smooth structures on } M\text{, fixed near } \partial M \}$$

("space" interpreted liberally).

Recording germs of smooth structure near each point gives a map

$$\mathcal{S}m(M) \longrightarrow \Gamma_\partial(\mathcal{S}m(TM) \to M)$$

(the space of sections of the bundle with fibre $\mathcal{S}m(T_mM) \cong \mathcal{S}m(\mathbb{R}^d)$)

Theorem. [Hirsch–Mazur '74, Kirby–Siebenmann '77]
For $d \neq 4$ this map is a homotopy equivalence.

Smoothing theory

M a topological d-manifold, maybe with smooth boundary ∂M

$$\mathcal{S}m(M) = \{ \text{ space of smooth structures on } M, \text{ fixed near } \partial M \}$$

(“space” interpreted liberally).

Recording germs of smooth structure near each point gives a map

$$\mathcal{S}m(M) \longrightarrow \Gamma_\partial(\mathcal{S}m(TM) \to M)$$

(the space of sections of the bundle with fibre $\mathcal{S}m(T_mM) \cong \mathcal{S}m(\mathbb{R}^d)$)

Theorem. [Hirsch–Mazur ’74, Kirby–Siebenmann ’77]
For $d \neq 4$ this map is a homotopy equivalence.

$Homeo_\partial(M)$ acts on $\mathcal{S}m(M)$, giving

$$\mathcal{S}m(M) \cong \coprod_{[W]} Homeo_\partial(W)/Diff_\partial(W)$$

Similarly, $\mathcal{S}m(\mathbb{R}^d) \cong Homeo(\mathbb{R}^d)/Diff(\mathbb{R}^d)$

A consequence of smoothing theory

Write $\mathit{Top}(d) := \mathit{Homeo}(\mathbb{R}^d)$. By linearising have $\mathit{Diff}(\mathbb{R}^d) \simeq O(d)$, so

$$\mathcal{S}m(\mathbb{R}^d) \simeq \mathit{Top}(d)/O(d).$$

A consequence of smoothing theory

Write $\mathit{Top}(d) := \mathit{Homeo}(\mathbb{R}^d)$. By linearising have $\mathit{Diff}(\mathbb{R}^d) \simeq O(d)$, so

$$\mathcal{S}m(\mathbb{R}^d) \simeq \mathit{Top}(d)/O(d).$$

Applied to D^d, $d \neq 4$, smoothing theory gives a map

$$\mathit{Homeo}_\partial(D^d)/\mathit{Diff}_\partial(D^d) \longrightarrow \Gamma_\partial(\mathcal{S}m(TD^d) \to D^d) = \mathit{map}_\partial(D^d, \mathit{Top}(d)/O(d))$$

which is a homotopy equivalence to the path components it hits.

A consequence of smoothing theory

Write $Top(d) := Homeo(\mathbb{R}^d)$. By linearising have $Diff(\mathbb{R}^d) \simeq O(d)$, so

$$\mathcal{S}m(\mathbb{R}^d) \simeq Top(d)/O(d).$$

Applied to D^d, $d \neq 4$, smoothing theory gives a map

$$Homeo_\partial(D^d)/Diff_\partial(D^d) \longrightarrow \Gamma_\partial(\mathcal{S}m(TD^d) \to D^d) = map_\partial(D^d, Top(d)/O(d))$$

which is a homotopy equivalence to the path components it hits.

The Alexander trick $Homeo_\partial(D^d) \simeq *$ implies

$$BDiff_\partial(D^d) \simeq \Omega^d_0 Top(d)/O(d) \qquad \text{(Morlet)}$$

or if you prefer

$$Diff_\partial(D^d) \simeq \Omega^{d+1} Top(d)/O(d).$$

A consequence of smoothing theory

Write $Top(d) := Homeo(\mathbb{R}^d)$. By linearising have $Diff(\mathbb{R}^d) \simeq O(d)$, so

$$\mathcal{S}m(\mathbb{R}^d) \simeq Top(d)/O(d).$$

Applied to D^d, $d \neq 4$, smoothing theory gives a map

$$Homeo_\partial(D^d)/Diff_\partial(D^d) \longrightarrow \Gamma_\partial(\mathcal{S}m(TD^d) \to D^d) = map_\partial(D^d, Top(d)/O(d))$$

which is a homotopy equivalence to the path components it hits.

The Alexander trick $Homeo_\partial(D^d) \simeq *$ implies

$$BDiff_\partial(D^d) \simeq \Omega^d_0 Top(d)/O(d) \qquad \text{(Morlet)}$$

or if you prefer

$$Diff_\partial(D^d) \simeq \Omega^{d+1} Top(d)/O(d).$$

$O(d)$ is "well understood" so $Diff_\partial(D^d)$ and $Top(d)$ are equidifficult.

But $Diff_\partial(D^d)$ is more approachable: can *use* smoothness.

What do we know?

The theorem of Farrell and Hsiang

The classical approach to studying $Diff_\partial(M)$ breaks up as

The theorem of Farrell and Hsiang

The classical approach to studying $Diff_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $hAut_\partial(M)$ analysed by homotopy theory.

The theorem of Farrell and Hsiang

The classical approach to studying $\mathit{Diff}_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $\mathit{hAut}_\partial(M)$ analysed by homotopy theory.
2. Comparison $\mathit{hAut}_\partial(M)/\widetilde{\mathit{Diff}}_\partial(M)$ with "block-diffeomorphisms" analysed by surgery theory.

The theorem of Farrell and Hsiang

The classical approach to studying $\mathit{Diff}_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $\mathit{hAut}_\partial(M)$
   analysed by homotopy theory.
2. Comparison $\mathit{hAut}_\partial(M)/\widetilde{\mathit{Diff}}_\partial(M)$ with "block-diffeomorphisms"
   analysed by surgery theory.
3. Comparison $\widetilde{\mathit{Diff}}_\partial(M)/\mathit{Diff}_\partial(M)$ with diffeomorphisms
   analysed by pseudoisotopy theory (and hence K-theory), but
   only valid in the "pseudoisotopy stable range".

The theorem of Farrell and Hsiang

The classical approach to studying $\mathit{Diff}_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $\mathit{hAut}_\partial(M)$ analysed by homotopy theory.
2. Comparison $\mathit{hAut}_\partial(M)/\widetilde{\mathit{Diff}}_\partial(M)$ with “block-diffeomorphisms” analysed by surgery theory.
3. Comparison $\widetilde{\mathit{Diff}}_\partial(M)/\mathit{Diff}_\partial(M)$ with diffeomorphisms analysed by pseudoisotopy theory (and hence K-theory), but only valid in the “pseudoisotopy stable range”.
   [Igusa ’84]: this is at least $min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$.

The theorem of Farrell and Hsiang

The classical approach to studying $Diff_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $hAut_\partial(M)$ analysed by homotopy theory.
2. Comparison $hAut_\partial(M)/\widetilde{Diff}_\partial(M)$ with "block-diffeomorphisms" analysed by surgery theory.
3. Comparison $\widetilde{Diff}_\partial(M)/Diff_\partial(M)$ with diffeomorphisms analysed by pseudoisotopy theory (and hence K-theory), but only valid in the "pseudoisotopy stable range".
   [Igusa '84]: this is at least $min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$.
   [RW '17]: it is at most $d-2$.

The theorem of Farrell and Hsiang

The classical approach to studying $Diff_\partial(M)$ breaks up as

1. Space of homotopy self-equivalences $hAut_\partial(M)$ analysed by homotopy theory.
2. Comparison $hAut_\partial(M)/\widetilde{Diff}_\partial(M)$ with "block-diffeomorphisms" analysed by surgery theory.
3. Comparison $\widetilde{Diff}_\partial(M)/Diff_\partial(M)$ with diffeomorphisms analysed by pseudoisotopy theory (and hence K-theory), but only valid in the "pseudoisotopy stable range".
   [Igusa '84]: this is at least $min(\frac{d-7}{2}, \frac{d-4}{3}) \sim \frac{d}{3}$.
   [RW '17]: it is at most $d-2$.

Theorem. [Farrell–Hsiang '78]

$$\pi_*(BDiff_\partial(D^d)) \otimes \mathbb{Q} = \begin{cases} 0 & d \text{ even} \\ \mathbb{Q}[4] \oplus \mathbb{Q}[8] \oplus \mathbb{Q}[12] \oplus \cdots & d \text{ odd} \end{cases}$$

in the pseudoisotopy stable range for d (so certainly for $* \lesssim \frac{d}{3}$).

The theorems of Watanabe

Theorem. [Watanabe '09]
For $2n + 1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_\partial(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

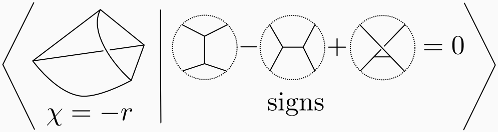
The theorems of Watanabe

Theorem. [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_\partial(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

where

$\mathcal{A}_r^{odd} =$



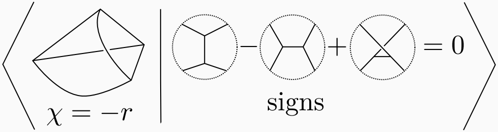
The theorems of Watanabe

Theorem. [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_\partial(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

where

$\mathcal{A}_r^{odd} =$



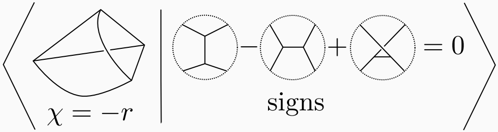
has $dim(\mathcal{A}_r^{odd}) = 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 9, \ldots$

Theorem. [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_\partial(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

where

$$\mathcal{A}_r^{odd} =$$



has $dim(\mathcal{A}_r^{odd}) = 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 9, \ldots$

Theorem. [Watanabe '18]
There is a surjection

$$\pi_r(BDiff_\partial(D^4)) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{even}$$

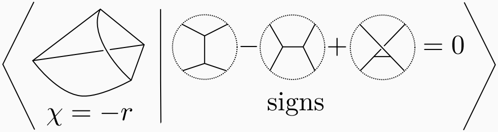
The theorems of Watanabe

Theorem. [Watanabe '09]
For $2n+1 \geq 5$ and $r \geq 2$ there is a surjection

$$\pi_{(2r)(2n)}(BDiff_\partial(D^{2n+1})) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{odd}$$

where

$$\mathcal{A}_r^{odd} =$$



has $dim(\mathcal{A}_r^{odd}) = 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 9, \ldots$

Theorem. [Watanabe '18]
There is a surjection

$$\pi_r(BDiff_\partial(D^4)) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_r^{even}$$

where $dim(\mathcal{A}_r^{even}) = 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$ (so $\pi_2(BDiff_\partial(D^4)) \neq 0$)

The theorem of Weiss

Closely related to the classical story is the fact that the stable map

$$O = \operatorname*{colim}_{d \to \infty} O(d) \longrightarrow Top = \operatorname*{colim}_{d \to \infty} Top(d)$$

is a $\mathbb{Q}$-equivalence, and hence

$$H^*(BTop; \mathbb{Q}) \cong H^*(BO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

The theorem of Weiss

Closely related to the classical story is the fact that the stable map

$$O = \operatorname*{colim}_{d \to \infty} O(d) \longrightarrow Top = \operatorname*{colim}_{d \to \infty} Top(d)$$

is a $\mathbb{Q}$-equivalence, and hence

$$H^*(BTop; \mathbb{Q}) \cong H^*(BO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

In $H^*(BO(2n); \mathbb{Q})$ the usual definition of Pontrjagin classes shows

$$p_n = e^2 \text{ and } p_{n+i} = 0 \text{ for all } i > 0. \qquad (!)$$

The theorem of Weiss

Closely related to the classical story is the fact that the stable map

$$O = \operatorname*{colim}_{d\to\infty} O(d) \longrightarrow Top = \operatorname*{colim}_{d\to\infty} Top(d)$$

is a $\mathbb{Q}$-equivalence, and hence

$$H^*(BTop;\mathbb{Q}) \cong H^*(BO;\mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

In $H^*(BO(2n);\mathbb{Q})$ the usual definition of Pontrjagin classes shows

$$p_n = e^2 \text{ and } p_{n+i} = 0 \text{ for all } i > 0. \tag{!}$$

Theorem. [Weiss '15]
For many n and $i \geq 0$ there are classes $w_{n,i} \in \pi_{4(n+i)}(BTop(2n))$ which pair nontrivially with p_{n+i} (i.e. (!) does not hold on $BTop(2n)$).

$\Rightarrow \pi_{2n-1+4i}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \neq 0$ for such n and i.

A pattern

A pattern

Inspired by Weiss' argument, Alexander Kupers and I have begun a programme to determine

$$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$$

as completely as possible. The first installment is:

A. Kupers, O. R-W, *On diffeomorphisms of even-dimensional discs* (arXiv:2007.13884)

A pattern

Inspired by Weiss' argument, Alexander Kupers and I have begun a programme to determine

$$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$$

as completely as possible. The first installment is:

A. Kupers, O. R-W, *On diffeomorphisms of even-dimensional discs* (arXiv:2007.13884)

Here we

1. fully determine these groups in degrees $* \leq 4n - 10$,

A pattern

Inspired by Weiss' argument, Alexander Kupers and I have begun a programme to determine

$$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$$

as completely as possible. The first installment is:

A. Kupers, O. R-W, *On diffeomorphisms of even-dimensional discs* (arXiv:2007.13884)

Here we

1. fully determine these groups in degrees $* \leq 4n - 10$,
2. determine them in higher degrees outside of certain "bands",

A pattern

Inspired by Weiss' argument, Alexander Kupers and I have begun a programme to determine

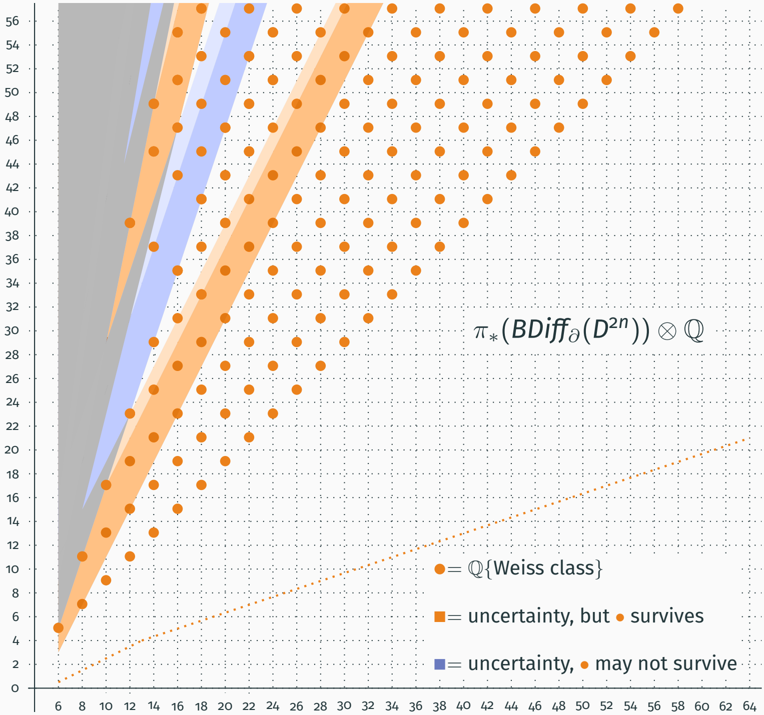
$$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$$

as completely as possible. The first installment is:

A. Kupers, O. R-W, *On diffeomorphisms of even-dimensional discs* (arXiv:2007.13884)

Here we

1. fully determine these groups in degrees $* \leq 4n - 10$,
2. determine them in higher degrees outside of certain "bands",
3. understand something about the structure of these bands.


$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$
= $\mathbb{Q}${Weiss class}
= uncertainty, but ● survives
= uncertainty, ● may not survive
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56
6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64

A pattern

Theorem. [Kupers–R-W]
Let $2n \geq 6$.

(i) If $d < 2n-1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ vanishes, and
(ii) if $d \geq 2n-1$ then $\pi_d(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$ is

$$\begin{cases} \mathbb{Q} & \text{if } d \equiv 2n{-}1 \bmod 4 \text{ and } d \notin \bigcup\limits_{r \geq 2} [2r(n{-}2)-1, 2rn-1], \\ 0 & \text{if } d \not\equiv 2n{-}1 \bmod 4 \text{ and } d \notin \bigcup\limits_{r \geq 2} [2r(n{-}2)-1, 2rn-1], \\ ? & \text{otherwise.} \end{cases}$$

A pattern

Using the fibre sequence $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

Reformulation (slightly stronger).
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

$$* \in \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-2].$$

A pattern

Using the fibre sequence $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

Reformulation (slightly stronger).
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

$$* \in \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-2].$$

Reflecting D^{2n} or $\mathbb{R}^{2n}$ induces compatible involutions on

$$\Omega_0^{2n+1}\frac{Top}{Top(2n)} \longrightarrow BDiff_\partial(D^{2n}) \simeq \Omega_0^{2n}\frac{Top(2n)}{O(2n)} \longrightarrow \Omega_0^{2n}\frac{Top}{O(2n)}.$$

A pattern

Using the fibre sequence $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

Reformulation (slightly stronger).
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

$$* \in \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-2].$$

Reflecting D^{2n} or $\mathbb{R}^{2n}$ induces compatible involutions on

$$\Omega_0^{2n+1} \frac{Top}{Top(2n)} \longrightarrow BDiff_\partial(D^{2n}) \simeq \Omega_0^{2n} \frac{Top(2n)}{O(2n)} \longrightarrow \Omega_0^{2n} \frac{Top}{O(2n)}.$$

We show this acts as -1 on

$$\pi_*(\Omega_0^{2n} \frac{Top}{O(2n)}) \otimes \mathbb{Q} = \mathbb{Q}[2n-1] \oplus \mathbb{Q}[2n+3] \oplus \mathbb{Q}[2n+7] \oplus \cdots$$

and acts on $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ as $(-1)^r$ in the rth band.

A pattern

Using the fibre sequence $\frac{Top(2n)}{O(2n)} \to \frac{Top}{O(2n)} \to \frac{Top}{Top(2n)}$ we have the

Reformulation (slightly stronger).
For $2n \geq 6$ the groups $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ are supported in degrees

$$* \in \bigcup_{r \geq 2} [2r(n-2)-1, 2rn-2].$$

Reflecting D^{2n} or $\mathbb{R}^{2n}$ induces compatible involutions on

$$\Omega_0^{2n+1}\frac{Top}{Top(2n)} \longrightarrow BDiff_\partial(D^{2n}) \simeq \Omega_0^{2n}\frac{Top(2n)}{O(2n)} \longrightarrow \Omega_0^{2n}\frac{Top}{O(2n)}.$$

We show this acts as -1 on

$$\pi_*(\Omega_0^{2n}\frac{Top}{O(2n)}) \otimes \mathbb{Q} = \mathbb{Q}[2n-1] \oplus \mathbb{Q}[2n+3] \oplus \mathbb{Q}[2n+7] \oplus \cdots$$

and acts on $\pi_*(\Omega_0^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ as $(-1)^r$ in the rth band.

The orange/blue colours in the chart are the $+1/-1$ eigenspaces.

The first uncertainty

We also determine to some extent what happens in the first band shown in the chart:

The first uncertainty

We also determine to some extent what happens in the first band shown in the chart: the groups $\pi_*(\Omega^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ in degrees $[4n-9, 4n-4]$ are calculated by a chain complex of the form

$$\mathbb{Q}^2 \longleftarrow \mathbb{Q}^4 \longleftarrow \mathbb{Q}^{10} \longleftarrow \mathbb{Q}^{21} \longleftarrow \mathbb{Q}^{15} \hookleftarrow \mathbb{Q}^3$$

The first uncertainty

We also determine to some extent what happens in the first band shown in the chart: the groups $\pi_*(\Omega^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ in degrees $[4n-9, 4n-4]$ are calculated by a chain complex of the form

$$\mathbb{Q}^2 \longleftarrow \mathbb{Q}^4 \longleftarrow \mathbb{Q}^{10} \longleftarrow \mathbb{Q}^{21} \longleftarrow \mathbb{Q}^{15} \hookleftarrow \mathbb{Q}^3$$

We don't know the differentials, but it has Euler characteristic 1 so must have some homology.

It lies in the $+1$-eigenspace, so injects into $\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$.

The first uncertainty

We also determine to some extent what happens in the first band shown in the chart: the groups $\pi_*(\Omega^{2n+1}(\frac{Top}{Top(2n)})) \otimes \mathbb{Q}$ in degrees $[4n-9, 4n-4]$ are calculated by a chain complex of the form

$$\mathbb{Q}^2 \longleftarrow \mathbb{Q}^4 \longleftarrow \mathbb{Q}^{10} \longleftarrow \mathbb{Q}^{21} \longleftarrow \mathbb{Q}^{15} \hookleftarrow \mathbb{Q}^3$$

We don't know the differentials, but it has Euler characteristic 1 so must have some homology.

It lies in the $+1$-eigenspace, so injects into $\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$.

By analogy with Watanabe's theorem for D^4 one expects

$$dim\, \pi_{4n-6}(BDiff_\partial(D^{2n})) \otimes \mathbb{Q} \geq 1$$

which is compatible with the above.

Details of the proof

Philosophy

Many results in this flavour of geometric topology are *relative*: they describe the difference between

1. topological/smooth manifolds (smoothing)
2. homotopy equivalences/block diffeomorphisms (surgery)
3. block diffeomorphisms/diffeomorphisms (pseudoisotopy)

Philosophy

Many results in this flavour of geometric topology are *relative*: they describe the difference between

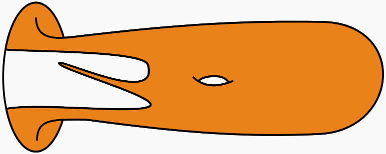
1. topological/smooth manifolds (smoothing)
2. homotopy equivalences/block diffeomorphisms (surgery)
3. block diffeomorphisms/diffeomorphisms (pseudoisotopy)

Weiss has suggested a new kind of relativisation:

for M with $\partial M = S^{d-1}$ and $\frac{1}{2}\partial M := D^{d-1} \subset S^{d-1}$ he showed that

$$\frac{Diff_\partial(M)}{Diff_\partial(D^d)} \simeq Emb^{\cong}_{1/2\partial}(M)$$

the space of self-embeddings of M relative to half its boundary, which are isotopic to diffeomorphisms.



Philosophy

$\Rightarrow$ Weiss fibre sequence

$$BDiff_\partial(D^d) \longrightarrow BDiff_\partial(M) \longrightarrow BEmb^{\cong}_{1/2\partial}(M)$$

Such a self-embedding space can often be analysed using the theory of embedding calculus.

Philosophy

$\Rightarrow$ Weiss fibre sequence

$$BDiff_\partial(D^d) \longrightarrow BDiff_\partial(M) \longrightarrow BEmb^{\cong}_{1/2\partial}(M)$$

Such a self-embedding space can often be analysed using the theory of embedding calculus.

Strategy: find a manifold M for which one can understand $BEmb^{\cong}_{1/2\partial}(M)$ and $BDiff_\partial(M)$, then deduce things about $BDiff_\partial(D^d)$.

Philosophy

$\Rightarrow$ Weiss fibre sequence

$$BDiff_\partial(D^d) \longrightarrow BDiff_\partial(M) \longrightarrow BEmb^{\cong}_{1/2\partial}(M)$$

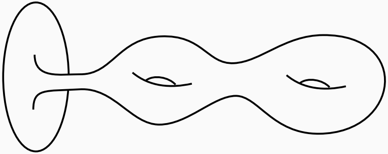
Such a self-embedding space can often be analysed using the theory of embedding calculus.

Strategy: find a manifold M for which one can understand $BEmb^{\cong}_{1/2\partial}(M)$ and $BDiff_\partial(M)$, then deduce things about $BDiff_\partial(D^d)$.

A good choice is

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$

especially for "arbitrarily large" g.



Philosophy

$\Rightarrow$ Weiss fibre sequence

$$BDiff_\partial(D^d) \longrightarrow BDiff_\partial(M) \longrightarrow BEmb^{\cong}_{1/2\partial}(M)$$

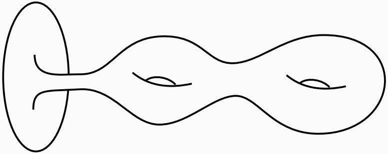
Such a self-embedding space can often be analysed using the theory of embedding calculus.

Strategy: find a manifold M for which one can understand $BEmb^{\cong}_{1/2\partial}(M)$ and $BDiff_\partial(M)$, then deduce things about $BDiff_\partial(D^d)$.

A good choice is

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$

especially for "arbitrarily large" g.



Theorem. [Madsen–Weiss '07 $2n = 2$, Galatius–R-W '14 $2n \geq 4$]

$$\lim_{g\to\infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$$

Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

Diffeomorphism groups

Embedding calculus (which will be discussed by A. Kupers in the next talk) will only allow us to access $\pi_*(BEmb^{\cong}_{1/2\partial}(W_{g,1})) \otimes \mathbb{Q}$, so to pursue the strategy requires $\pi_*(BDiff_\partial(W_{g,1})) \otimes \mathbb{Q}$ instead of $H^*(BDiff_\partial(W_{g,1}); \mathbb{Q})$.

Diffeomorphism groups

Embedding calculus (which will be discussed by A. Kupers in the next talk) will only allow us to access $\pi_*(BEmb^{\cong}_{1/2\partial}(W_{g,1})) \otimes \mathbb{Q}$, so to pursue the strategy requires $\pi_*(BDiff_\partial(W_{g,1})) \otimes \mathbb{Q}$ instead of $H^*(BDiff_\partial(W_{g,1}); \mathbb{Q})$.

Theorem. [Kreck '79]
For $n \geq 3$ there are extensions

$$0 \longrightarrow I_g \longrightarrow \pi_1(BDiff_\partial(W_{g,1})) \longrightarrow \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even} \\ Sp_{2g}(\mathbb{Z}) & n = 3, 7 \\ Sp^q_{2g}(\mathbb{Z}) & n \text{ odd not } 3,7 \end{cases}$$

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow I_g \longrightarrow Hom(H_n(W_{g,1}; \mathbb{Z}), S\pi_n SO(n)) \longrightarrow 0.$$

Diffeomorphism groups

Embedding calculus (which will be discussed by A. Kupers in the next talk) will only allow us to access $\pi_*(BEmb^{\cong}_{1/2\partial}(W_{g,1})) \otimes \mathbb{Q}$, so to pursue the strategy requires $\pi_*(BDiff_\partial(W_{g,1})) \otimes \mathbb{Q}$ instead of $H^*(BDiff_\partial(W_{g,1}); \mathbb{Q})$.

Theorem. [Kreck '79]
For $n \geq 3$ there are extensions

$$0 \longrightarrow I_g \longrightarrow \pi_1(BDiff_\partial(W_{g,1})) \longrightarrow \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even} \\ Sp_{2g}(\mathbb{Z}) & n = 3, 7 \\ Sp^q_{2g}(\mathbb{Z}) & n \text{ odd not 3,7} \end{cases}$$

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow I_g \longrightarrow Hom(H_n(W_{g,1}; \mathbb{Z}), S\pi_n SO(n)) \longrightarrow 0.$$

$\Rightarrow \pi_1(BDiff_\partial(W_{g,1}))$ wildly complicated group, not nilpotent: cannot expect to determine the rational homotopy of $BDiff_\partial(W_{g,1})$ from its rational cohomology.

Torelli groups

Can pass to the (infinite index) Torelli subgroup

$$Tor_\partial(W_{g,1}) := ker\left(Diff_\partial(W_{g,1}) \twoheadrightarrow G'_g := \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even} \\ Sp_{2g}(\mathbb{Z}) & n = 3, 7 \\ Sp^q_{2g}(\mathbb{Z}) & n \text{ odd not 3,7} \end{cases} \right)$$

to eliminate the arithmetic group, but this changes the cohomology.

Torelli groups

Can pass to the (infinite index) Torelli subgroup

$$Tor_\partial(W_{g,1}) := ker\left(Diff_\partial(W_{g,1}) \twoheadrightarrow G'_g := \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even} \\ Sp_{2g}(\mathbb{Z}) & n = 3, 7 \\ Sp^q_{2g}(\mathbb{Z}) & n \text{ odd not } 3{,}7 \end{cases}\right)$$

to eliminate the arithmetic group, but this changes the cohomology.

In A. Kupers, O. R-W, *The cohomology of Torelli groups is algebraic*
Forum of Mathematics, Sigma, to appear

(i) $BTor_\partial(W_{g,1})$ is nilpotent,
(ii) G'_g-representations $H^i(BTor_\partial(W_{g,1}); \mathbb{Q})$ are *algebraic*, i.e. extend to representations of the ambient algebraic groups $\mathbf{O}_{g,g}$ or $\mathbf{Sp}_{2g}$.

Torelli groups

Can pass to the (infinite index) Torelli subgroup

$$Tor_\partial(W_{g,1}) := ker\left(Diff_\partial(W_{g,1}) \twoheadrightarrow G'_g := \begin{cases} O_{g,g}(\mathbb{Z}) & n \text{ even} \\ Sp_{2g}(\mathbb{Z}) & n = 3, 7 \\ Sp^q_{2g}(\mathbb{Z}) & n \text{ odd not } 3,7 \end{cases} \right)$$

to eliminate the arithmetic group, but this changes the cohomology.

In A. Kupers, O. R-W, *The cohomology of Torelli groups is algebraic*
Forum of Mathematics, Sigma, to appear

(i) $BTor_\partial(W_{g,1})$ is nilpotent,
(ii) G'_g-representations $H^i(BTor_\partial(W_{g,1}); \mathbb{Q})$ are *algebraic*, i.e. extend to representations of the ambient algebraic groups $\mathbf{O}_{g,g}$ or $\mathbf{Sp}_{2g}$.

This is done using the Torelli version of the Weiss fibre sequence

$$BDiff_\partial(D^{2n}) \longrightarrow BTor_\partial(W_{g,1}) \longrightarrow BTorEmb^{\cong}_{1/2\partial}(W_{g,1})$$

and embedding calculus to *qualitatively* understand the third term; the first contributes only trivial G'_g-representations.

(Twisted) Miller–Morita–Mumford classes

The space $BTor_\partial(W_{g,1})$ carries a smooth bundle

$$W_g \xrightarrow{i} E \xrightarrow{\pi} BTor_\partial(W_{g,1})$$

($W_g = \#^g S^n \times S^n$) with a trivial sub-$D^{2n}$-bundle and a trivialisation of the local system $\mathcal{H}^n(W_g; \mathbb{Z})$.

(Twisted) Miller–Morita–Mumford classes

The space $BTor_\partial(W_{g,1})$ carries a smooth bundle

$$W_g \xrightarrow{i} E \xrightarrow{\pi} BTor_\partial(W_{g,1})$$

($W_g = \#^g S^n \times S^n$) with a trivial sub-$D^{2n}$-bundle and a trivialisation of the local system $\mathcal{H}^n(W_g; \mathbb{Z})$.

It has a section s given by $0 \in D^{2n}$, so a split exact sequence

$$0 \longrightarrow H^n(BTor_\partial(W_{g,1}); \mathbb{Q}) \xrightarrow{\pi^*} H^n(E; \mathbb{Q}) \xrightarrow{i^*} H^n(W_g; \mathbb{Q}) \longrightarrow 0$$

with splitting $\iota : H^n(W_g; \mathbb{Q}) \to H^n(E; \mathbb{Q})$.

(Twisted) Miller–Morita–Mumford classes

The space $BTor_\partial(W_{g,1})$ carries a smooth bundle

$$W_g \xrightarrow{i} E \xrightarrow{\pi} BTor_\partial(W_{g,1})$$

($W_g = \#^g S^n \times S^n$) with a trivial sub-$D^{2n}$-bundle and a trivialisation of the local system $\mathcal{H}^n(W_g;\mathbb{Z})$.

It has a section s given by $0 \in D^{2n}$, so a split exact sequence

$$0 \longrightarrow H^n(BTor_\partial(W_{g,1});\mathbb{Q}) \xrightarrow{\pi^*} H^n(E;\mathbb{Q}) \xrightarrow{i^*} H^n(W_g;\mathbb{Q}) \longrightarrow 0$$

with splitting $\iota : H^n(W_g;\mathbb{Q}) \to H^n(E;\mathbb{Q})$.

Have vertical tangent bundle $T_\pi E \to E$, so for any $c \in H^*(BSO(2n);\mathbb{Q})$ and $v_1, \ldots, v_r \in H^n(W_g;\mathbb{Q})$ we can form

$$\kappa_c(v_1,\ldots,v_r) := \int_\pi c(T_\pi E)\cdot \iota(v_1)\cdots\iota(v_r) \in H^{|c|+n(r-2)}(BTor_\partial(W_{g,1});\mathbb{Q}).$$

(Twisted) Miller–Morita–Mumford classes

The space $BTor_\partial(W_{g,1})$ carries a smooth bundle

$$W_g \xrightarrow{i} E \xrightarrow{\pi} BTor_\partial(W_{g,1})$$

($W_g = \#^g S^n \times S^n$) with a trivial sub-$D^{2n}$-bundle and a trivialisation of the local system $\mathcal{H}^n(W_g; \mathbb{Z})$.

It has a section s given by $0 \in D^{2n}$, so a split exact sequence

$$0 \longrightarrow H^n(BTor_\partial(W_{g,1}); \mathbb{Q}) \xrightarrow{\pi^*} H^n(E; \mathbb{Q}) \xrightarrow{i^*} H^n(W_g; \mathbb{Q}) \longrightarrow 0$$

with splitting $\iota : H^n(W_g; \mathbb{Q}) \to H^n(E; \mathbb{Q})$.

Have vertical tangent bundle $T_\pi E \to E$, so for any $c \in H^*(BSO(2n); \mathbb{Q})$ and $v_1, \ldots, v_r \in H^n(W_g; \mathbb{Q})$ we can form

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r) \in H^{|c|+n(r-2)}(BTor_\partial(W_{g,1}); \mathbb{Q}).$$

Under the G'_g-action these transform via $G'_g \circlearrowright H^n(W_g; \mathbb{Q})$.

For $r = 0$ these are the usual Miller–Morita–Mumford classes.

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

(i) $(c, v_1, \ldots, v_r) \to \kappa_c(v_1, \ldots, v_r)$ is linear in c and each v_i,

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

(i) $(c, v_1, \ldots, v_r) \to \kappa_c(v_1, \ldots, v_r)$ is linear in c and each v_i,

(ii) $\kappa_c(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa_c(v_1, \ldots, v_r)$,

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \dots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

(i) $(c, v_1, \dots, v_r) \to \kappa_c(v_1, \dots, v_r)$ is linear in c and each v_i,

(ii) $\kappa_c(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa_c(v_1, \dots, v_r)$,

(iii) $\sum_i \kappa_c(v_1, \dots, v_j, a_i) \cdot \kappa_{c'}(a_i^\#, v_{j+1}, \dots, v_r) = \kappa_{c \cdot c'}(v_1, \dots, v_r)$,

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \dots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

(i) $(c, v_1, \dots, v_r) \to \kappa_c(v_1, \dots, v_r)$ is linear in c and each v_i,

(ii) $\kappa_c(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa_c(v_1, \dots, v_r)$,

(iii) $\sum_i \kappa_c(v_1, \dots, v_j, a_i) \cdot \kappa_{c'}(a_i^\#, v_{j+1}, \dots, v_r) = \kappa_{c \cdot c'}(v_1, \dots, v_r)$,

(iv) $\sum_i \kappa_c(v_1, \dots, v_r, a_i, a_i^\#) = \kappa_{e \cdot c}(v_1, \dots, v_r)$,

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

(i) $(c, v_1, \ldots, v_r) \to \kappa_c(v_1, \ldots, v_r)$ is linear in c and each v_i,

(ii) $\kappa_c(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa_c(v_1, \ldots, v_r)$,

(iii) $\sum_i \kappa_c(v_1, \ldots, v_j, a_i) \cdot \kappa_{c'}(a_i^\#, v_{j+1}, \ldots, v_r) = \kappa_{c \cdot c'}(v_1, \ldots, v_r)$,

(iv) $\sum_i \kappa_c(v_1, \ldots, v_r, a_i, a_i^\#) = \kappa_{e \cdot c}(v_1, \ldots, v_r)$,

(v) $\kappa_{\mathcal{L}_i} = 0$.

Relations among Twisted Miller–Morita–Mumford classes

$$\kappa_c(v_1, \ldots, v_r) := \int_\pi c(T_\pi E) \cdot \iota(v_1) \cdots \iota(v_r)$$

Let $\{a_i\}$ be a basis of $H^n(W_g; \mathbb{Q})$, and $\{a_i^\#\}$ be the Poincaré dual basis characterised by $\int_{W_g} a_i^\# \cdot a_j = \delta_{ij}$. It is easy to show that:

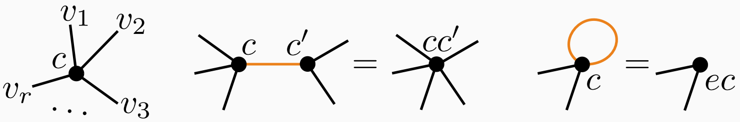
(i) $(c, v_1, \ldots, v_r) \to \kappa_c(v_1, \ldots, v_r)$ is linear in c and each v_i,

(ii) $\kappa_c(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = sign(\sigma)^n \cdot \kappa_c(v_1, \ldots, v_r)$,

(iii) $\sum_i \kappa_c(v_1, \ldots, v_j, a_i) \cdot \kappa_{c'}(a_i^\#, v_{j+1}, \ldots, v_r) = \kappa_{c \cdot c'}(v_1, \ldots, v_r)$,

(iv) $\sum_i \kappa_c(v_1, \ldots, v_r, a_i, a_i^\#) = \kappa_{e \cdot c}(v_1, \ldots, v_r)$,

(v) $\kappa_{\mathcal{L}_i} = 0$.



In A. Kupers, O. R-W, *On the cohomology of Torelli groups*
Forum of Mathematics, Pi, 8 (2020)

(combined with the algebraicity theorem) we show

Theorem. [Kupers–R-W '20]
If $2n \geq 6$ then the G'_g-equivariant ring homomorphism

$$\frac{\Lambda^*_{\mathbb{Q}}[\kappa_c(v_1, \ldots, v_r) \mid |c| + n(r-2) > 0]}{\text{(the relations (i) – (v))}} \longrightarrow H^*(BTor_\partial(W_{g,1}); \mathbb{Q})$$

is an isomorphism in a stable range.

Cohomology of Torelli groups

In A. Kupers, O. R-W, *On the cohomology of Torelli groups*
Forum of Mathematics, Pi, 8 (2020)

(combined with the algebraicity theorem) we show

Theorem. [Kupers–R-W '20]
If $2n \geq 6$ then the G'_g-equivariant ring homomorphism

$$\frac{\Lambda^*_{\mathbb{Q}}[\kappa_c(v_1, \ldots, v_r) \,|\, |c| + n(r-2) > 0]}{\text{(the relations (i) – (v))}} \longrightarrow H^*(BTor_\partial(W_{g,1}); \mathbb{Q})$$

is an isomorphism in a stable range.

Remark. This is not an efficient presentation! It can be simplified.

Cohomology of Torelli groups

In A. Kupers, O. R-W, *On the cohomology of Torelli groups*
Forum of Mathematics, Pi, 8 (2020)

(combined with the algebraicity theorem) we show

Theorem. [Kupers–R-W '20]
If $2n \geq 6$ then the G'_g-equivariant ring homomorphism

$$\frac{\Lambda^*_{\mathbb{Q}}[\kappa_c(v_1, \ldots, v_r) \,|\, |c| + n(r-2) > 0]}{\text{(the relations (i) – (v))}} \longrightarrow H^*(BTor_\partial(W_{g,1}); \mathbb{Q})$$

is an isomorphism in a stable range.

Remark. This is not an efficient presentation! It can be simplified.

Strategy: Every irreducible representation of $\mathbf{G}_g \in \{\mathbf{O}_{g,g}, \mathbf{Sp}_{2g}\}$ is a summand of $H^n(W_g;\mathbb{Q})^{\otimes k}$ for some k, so a map φ of algebraic $\mathbf{G}_g$-representations is an isomorphism $\Leftrightarrow [\varphi \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g}$ is for all k.

$\Rightarrow$ Evaluate $[- \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g}$ of both sides.

An idea of the proof I

Consider Serre spectral sequence with $\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k}$-coefficients for

$$BTor_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BG'_g.$$

An idea of the proof I

Consider Serre spectral sequence with $\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k}$-coefficients for

$$BTor_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BG'_g.$$

Using work of Borel on stable cohomology of arithmetic groups, it the form

$$\begin{aligned} E_2^{p,q} = H^p(\mathbf{G}_\infty;\mathbb{Q}) \otimes [H^q(BTor_\partial(W_{g,1});\mathbb{Q}) \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g} \\ \Rightarrow H^{p+q}(BDiff_\partial(W_{g,1});\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k}) \end{aligned}$$

in a stable range.

An idea of the proof I

Consider Serre spectral sequence with $\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k}$-coefficients for

$$BTor_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BG'_g.$$

Using work of Borel on stable cohomology of arithmetic groups, it the form

$$E_2^{p,q} = H^p(\mathbf{G}_\infty;\mathbb{Q}) \otimes [H^q(BTor_\partial(W_{g,1});\mathbb{Q}) \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g}$$
$$\Rightarrow H^{p+q}(BDiff_\partial(W_{g,1});\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k})$$

in a stable range.

Borel has calculated $H^*(\mathbf{G}_\infty;\mathbb{Q})$, and the work of Galatius–R-W can be used to calculate $H^*(BDiff_\partial(W_{g,1});\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k})$ in a stable range.

An idea of the proof I

Consider Serre spectral sequence with $\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k}$-coefficients for

$$BTor_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BG'_g.$$

Using work of Borel on stable cohomology of arithmetic groups, it the form

$$E_2^{p,q} = H^p(\mathbf{G}_\infty;\mathbb{Q}) \otimes [H^q(BTor_\partial(W_{g,1});\mathbb{Q}) \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g} \\ \Rightarrow H^{p+q}(BDiff_\partial(W_{g,1});\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k})$$

in a stable range.

Borel has calculated $H^*(\mathbf{G}_\infty;\mathbb{Q})$, and the work of Galatius–R-W can be used to calculate $H^*(BDiff_\partial(W_{g,1});\mathcal{H}^n(W_g;\mathbb{Q})^{\otimes k})$ in a stable range.

$\Rightarrow$ collapse and determines $[H^*(BTor_\partial(W_{g,1});\mathbb{Q}) \otimes H^n(W_g;\mathbb{Q})^{\otimes k}]^{\mathbf{G}_g}$ to be given by partitions of $\{1, 2, \ldots, k\}$ with parts labelled by $\mathcal{B}$ (with some constraints on degrees of labels).

An idea of the proof II

Classical invariant theory shows that

$$[H^n(W_g;\mathbb{Q})^{\otimes r}]^{\mathbf{G}_g} \cong \{\text{(signed) matchings of } \{1,2,\ldots,r\}\}$$

for $g \gg r$; the bijection is implemented by inserting the invariant vector $\omega := \sum_i a_i \otimes a_i^{\#} \in H^n(W_g;\mathbb{Q})^{\otimes 2}$.

An idea of the proof II

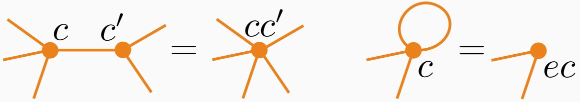
Classical invariant theory shows that

$$[H^n(W_g;\mathbb{Q})^{\otimes r}]^{\mathbf{G}_g} \cong \{\text{(signed) matchings of } \{1,2,\ldots,r\}\}$$

for $g \gg r$; the bijection is implemented by inserting the invariant vector $\omega := \sum_i a_i \otimes a_i^{\#} \in H^n(W_g;\mathbb{Q})^{\otimes 2}$.

$$\Rightarrow \left[\frac{\Lambda^*_{\mathbb{Q}}[\kappa_c(v_1,\ldots,v_r) \mid |c| + n(r-2) > 0]}{\text{(the relations (i) – (v))}} \otimes H^n(W_g;\mathbb{Q})^{\otimes k}\right]^{\mathbf{G}_g} =$$

the graded vector space of graphs with legs $\{1,2,\ldots,k\}$, vertices labelled by $c \in \mathcal{B}$, and certain signs I shall ignore, modulo



An idea of the proof II

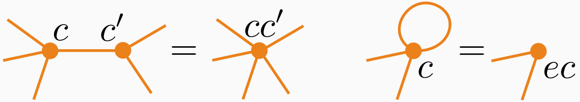
Classical invariant theory shows that

$$[H^n(W_g;\mathbb{Q})^{\otimes r}]^{\mathbf{G}_g} \cong \{\text{(signed) matchings of } \{1,2,\ldots,r\}\}$$

for $g \gg r$; the bijection is implemented by inserting the invariant vector $\omega := \sum_i a_i \otimes a_i^\# \in H^n(W_g;\mathbb{Q})^{\otimes 2}$.

$$\Rightarrow \left[\frac{\Lambda^*_{\mathbb{Q}}[\kappa_c(v_1,\ldots,v_r) \mid |c| + n(r-2) > 0]}{\text{(the relations (i) – (v))}} \otimes H^n(W_g;\mathbb{Q})^{\otimes k}\right]^{\mathbf{G}_g} =$$

the graded vector space of graphs with legs $\{1,2,\ldots,k\}$, vertices labelled by $c \in \mathcal{B}$, and certain signs I shall ignore, modulo



Using these relations to contract all internal edges, this is the same as partitions of $\{1,2,\ldots,k\}$ with parts labelled by $\mathcal{B}$ (with some constraints). □

Returning to the disc

To prove our results about $BDiff_\partial(D^{2n})$ we in fact work with the *framed* analogue of the Weiss fibre sequence

$$BDiff^{fr}_\partial(D^{2n}) \longrightarrow BDiff^{fr}_\partial(W_{g,1}) \longrightarrow BEmb^{\cong,fr}_{1/2\partial}(W_{g,1}).$$

Returning to the disc

To prove our results about $BDiff_\partial(D^{2n})$ we in fact work with the *framed* analogue of the Weiss fibre sequence

$$BDiff_\partial^{fr}(D^{2n}) \longrightarrow BDiff_\partial^{fr}(W_{g,1}) \longrightarrow BEmb_{1/2\partial}^{\cong,fr}(W_{g,1}).$$

The story is more complicated in the framed case. Have a fibration

$$X_1(g) \longrightarrow BTor_\partial^{fr}(W_{g,1}) \longrightarrow X_0$$

with $H^*(X_0;\mathbb{Q}) = \Lambda_{\mathbb{Q}}[\bar{\sigma}_{4j-2n-1} \,|\, j > n/2]$.

Returning to the disc

To prove our results about $BDiff_\partial(D^{2n})$ we in fact work with the *framed* analogue of the Weiss fibre sequence

$$BDiff^{fr}_\partial(D^{2n}) \longrightarrow BDiff^{fr}_\partial(W_{g,1}) \longrightarrow BEmb^{\cong,fr}_{1/2\partial}(W_{g,1}).$$

The story is more complicated in the framed case. Have a fibration

$$X_1(g) \longrightarrow BTor^{fr}_\partial(W_{g,1}) \longrightarrow X_0$$

with $H^*(X_0;\mathbb{Q}) = \Lambda_\mathbb{Q}[\bar{\sigma}_{4j-2n-1} \,|\, j > n/2]$.

We show that in a stable range $H^*(X_1(g);\mathbb{Q})$ is generated by classes

$$\kappa(v_1, v_2, v_3) \in H^n(X_1(g);\mathbb{Q}) \qquad v_i \in H^n(W_{g,1};\mathbb{Q})$$

subject only to the relations

(i) linearity in each v_i,
(ii) $\kappa(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = sign(\sigma)^n \cdot \kappa(v_1, v_2, v_3)$,
(iii) $\sum_i \kappa(v_1, v_2, a_i) \cdot \kappa(a_i^\#, v_5, v_6) = \sum_i \kappa(v_1, v_5, a_i) \cdot \kappa(a_i^\#, v_6, v_2)$,
(iv) $\sum_i \kappa(v_1, a_i, a_i^\#) = 0$ for any v_1.

Returning to the disc

To prove our results about $BDiff_\partial(D^{2n})$ we in fact work with the *framed* analogue of the Weiss fibre sequence

$$BDiff^{fr}_\partial(D^{2n}) \longrightarrow BDiff^{fr}_\partial(W_{g,1}) \longrightarrow BEmb^{\cong,fr}_{1/2\partial}(W_{g,1}).$$

The story is more complicated in the framed case. Have a fibration

$$X_1(g) \longrightarrow BTor^{fr}_\partial(W_{g,1}) \longrightarrow X_0$$

with $H^*(X_0;\mathbb{Q}) = \Lambda_\mathbb{Q}[\bar{\sigma}_{4j-2n-1} \,|\, j > n/2]$.

We show that in a stable range $H^*(X_1(g);\mathbb{Q})$ is generated by classes

$$\kappa(v_1, v_2, v_3) \in H^n(X_1(g);\mathbb{Q}) \qquad v_i \in H^n(W_{g,1};\mathbb{Q})$$

subject only to the relations

(i) linearity in each v_i,

(ii) $\kappa(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = sign(\sigma)^n \cdot \kappa(v_1, v_2, v_3)$,

(iii) $\sum_i \kappa(v_1, v_2, a_i) \cdot \kappa(a_i^\#, v_5, v_6) = \sum_i \kappa(v_1, v_5, a_i) \cdot \kappa(a_i^\#, v_6, v_2)$,

(iv) $\sum_i \kappa(v_1, a_i, a_i^\#) = 0$ for any v_1.

Cohomology supported in degrees which are multiples of n.

Returning to the disc

The unstable Adams spectral sequence then shows

$$\pi_*(BTor^{fr}_\partial(W_{g,1}))\otimes\mathbb{Q} = \left(\bigoplus_{j>n/2}\mathbb{Q}[4j-2n-1]\right) \text{ “}\oplus\text{” } \left(\substack{\text{something supported in}\\ *\in\bigcup_{r\geq 0}[r(n-1)+1,rn-2]}\right)$$

Returning to the disc

The unstable Adams spectral sequence then shows

$$\pi_*(BTor^{fr}_\partial(W_{g,1}))\otimes\mathbb{Q} = \left(\bigoplus_{j>n/2} \mathbb{Q}[4j-2n-1]\right) \text{“}\oplus\text{”} \left(\substack{\text{something supported in} \\ *\in\bigcup_{r\geq 0}[r(n-1)+1, rn-2]}\right)$$

In the Torelli version of the framed Weiss fibre sequence

$$BDiff^{fr}_\partial(D^{2n}) \longrightarrow BTor^{fr}_\partial(W_{g,1}) \longrightarrow BTorEmb^{\cong,fr}_{1/2\partial}(W_{g,1})$$

the first part, coming from X_0, provides the Weiss classes, and the second part, coming from $X_1(g)$, provides the lightly-shaded unknown region in the chart.

Returning to the disc

The unstable Adams spectral sequence then shows

$$\pi_*(BTor^{fr}_\partial(W_{g,1}))\otimes\mathbb{Q} = \left(\bigoplus_{j>n/2} \mathbb{Q}[4j-2n-1]\right) \text{“}\oplus\text{”} \begin{pmatrix}\text{something supported in}\\ *\in\bigcup_{r\geq 0}[r(n-1)+1, rn-2]\end{pmatrix}$$

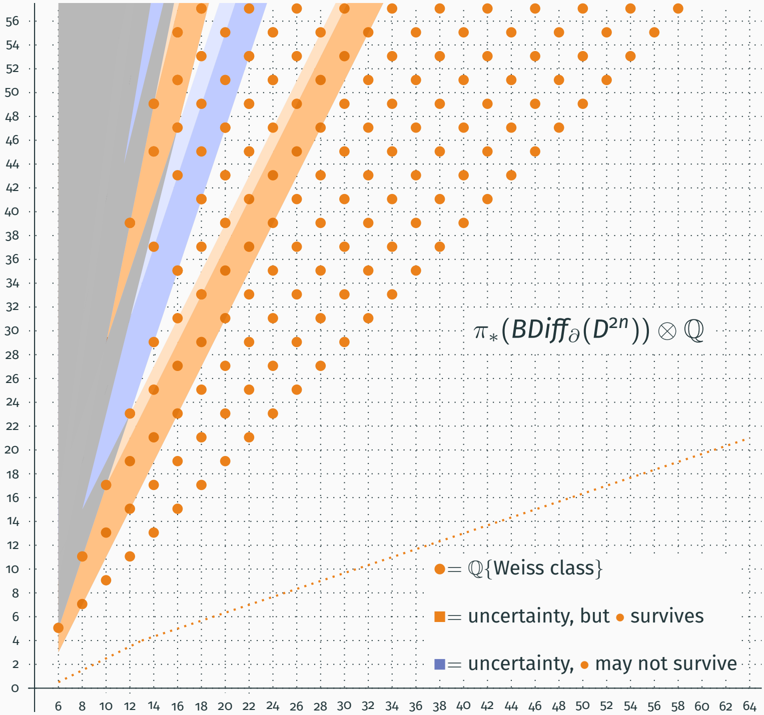
In the Torelli version of the framed Weiss fibre sequence

$$BDiff^{fr}_\partial(D^{2n}) \longrightarrow BTor^{fr}_\partial(W_{g,1}) \longrightarrow BTorEmb^{\cong,fr}_{1/2\partial}(W_{g,1})$$

the first part, coming from X_0, provides the Weiss classes, and the second part, coming from $X_1(g)$, provides the lightly-shaded unknown region in the chart.

In the next talk Alexander Kupers will explain the darkly-shaded unknown region, coming from analysing $\pi_*(BEmb^{\cong,fr}_{1/2\partial}(W_{g,1}))\otimes\mathbb{Q}$ via embedding calculus.

Questions?


$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$
● = $\mathbb{Q}$\{Weiss class\}
■ = uncertainty, but ● survives
■ = uncertainty, ● may not survive