Diffeomorphisms of discs

Oscar Randal-Williams

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European Research Council
Established by the European Commission

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Goal

$$
\begin{aligned}
\mathit{Diff}_\partial(D^d) &= \left\{ f : D^d \to D^d \;\middle|\; \begin{array}{c} f \text{ is a diffeomorphism which agrees} \\ \text{with the identity near } \partial D^d \end{array} \right\} \\
&\cong \left\{ f : \mathbb{R}^d \to \mathbb{R}^d \;\middle|\; f = \mathrm{Id}_{\mathbb{R}^d} \text{ outside a compact set} \right\}
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$\Rightarrow$ $\mathit{Diff}_\partial(D^d)$ measures how the full group of smooth symmetries of the d-sphere differs from the obvious linear symmetries.

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What is it then?

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I want to explain recent progress towards understanding the rational homotopy type of the classifying space

$$BDiff_{\partial}(D^d) \simeq \left\{ \begin{matrix} \text{Riemannian metrics on } D^d \\ \text{which are standard near } \partial D^d \end{matrix} \right\} / Diff_{\partial}(D^d).$$

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Everything new I will say represents collaborations with

Manuel Krannich and with Alexander Kupers

Four short stories

Homeomorphisms of $\mathbb{R}^d$

Smoothing theory says that for $d \neq 4$ we have

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$\Rightarrow$ understanding $BDiff_\partial(D^d) \sim$ understanding $Homeo(\mathbb{R}^d)$

Surgery and pseudoisotopy: Farrell and Hsiang

Classical strategy to study $\mathit{Diff}_\partial(M)$:

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“Configuration space integrals” $\rightsquigarrow$ classes in $H^*(BDiff_\partial^{fr}(D^d); \mathbb{Q})$

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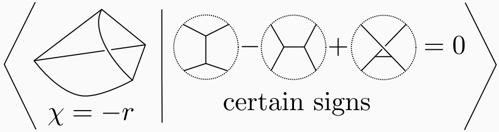
Theorem. [Watanabe ’09, ’18]

For d even, or d odd and $r > 1$, there is a surjection

$$\pi_{r\cdot(d-3)}(BDiff_\partial(D^d))_{\mathbb{Q}} \longrightarrow \mathcal{A}_r^{(-1)^d}$$

onto a certain vector space of trivalent graphs

$\mathcal{A}_r^{\pm} =$



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Topological Pontrjagin classes: Weiss

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Theorem. [Weiss '22]
For many n and $i \geq 0$ there exist

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$\Rightarrow \pi_{2n-1+4i}(BDiff_\partial(D^{2n}))_\mathbb{Q} \neq 0$, $\pi_{2n-2+4i}(BDiff_\partial(D^{2n+1}))_\mathbb{Q} \neq 0$

A pattern

A pattern: even-dimensional discs

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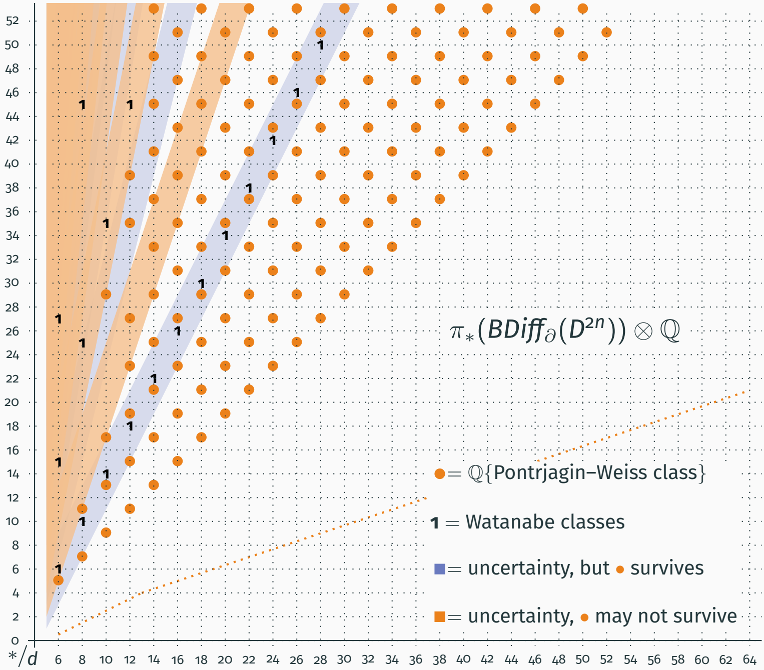
Theorem. [Kupers–R-W '20 '21]

Let $2n \geq 6$.

(i) If $i < 2n-1$ then $\pi_i(BDiff_\partial(D^{2n}))_\mathbb{Q}$ vanishes, and
(ii) if $i \geq 2n-1$ then $\pi_i(BDiff_\partial(D^{2n}))_\mathbb{Q}$ is

$$\begin{cases} \mathbb{Q} & \text{if } i \equiv 2n{-}1 \bmod 4 \text{ and } i \notin \bigcup_{r\geq 2} [2r(n{-}2)-1, 2r(n{-}1)+1], \\ 0 & \text{if } i \not\equiv 2n{-}1 \bmod 4 \text{ and } i \notin \bigcup_{r\geq 2} [2r(n{-}2)-1, 2r(n{-}1)+1], \\ ? & \text{otherwise.} \end{cases}$$

The $\mathbb{Q}$'s are all Pontrjagin–Weiss classes.



$\pi_*(BDiff_\partial(D^{2n})) \otimes \mathbb{Q}$
● = $\mathbb{Q}${Pontrjagin–Weiss class}
1 = Watanabe classes
■ = uncertainty, but ● survives
■ = uncertainty, ● may not survive
*/d

A pattern: odd-dimensional discs

Using different techniques, Manuel Krannich and I investigated

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Theorem. [Krannich–R-W ’21]

In degrees $i \leq 3n - 8$ we have

$$\pi_i(BDiff_\partial(D^{2n+1}))_\mathbb{Q} = K_{i+1}(\mathbb{Z})_\mathbb{Q} \oplus \begin{cases} \mathbb{Q} & i \equiv 2n-2 \mod 4, i \geq 2n-2 \\ 0 & \text{else} \end{cases}$$

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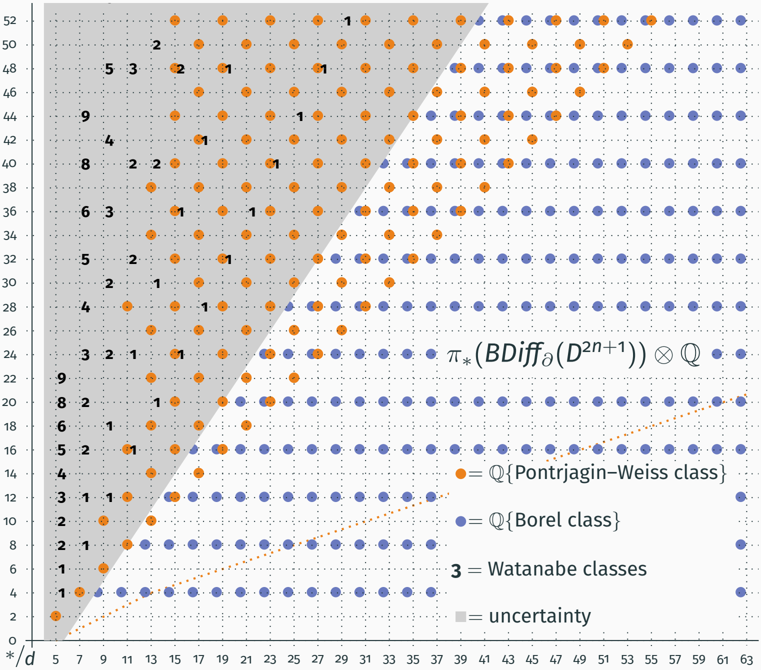
The one of degree $2n - 2$ is also the simplest of Watanabe’s classes, corresponding to the trivalent graph .



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= $\mathbb{Q}$\{Pontrjagin–Weiss class\}
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$*/d$

Outline of the method

Many results in this flavour of geometric topology are *relative*: they describe the difference between

1. topological/smooth manifolds (smoothing)
2. homotopy equivalences/block diffeomorphisms (surgery)
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Weiss introduced a powerful new kind of relativisation:

For M with $\partial M = S^{d-1}$ and $\frac{1}{2}\partial M := D^{d-1} \subset S^{d-1}$ he showed there is a homotopy fibre sequence

$$BDiff_{\partial}(D^d) \longrightarrow BDiff_{\partial}(M) \longrightarrow BEmb^{\cong}_{1/2\partial}(M)$$

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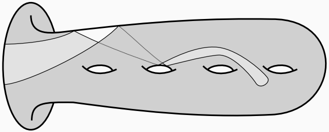

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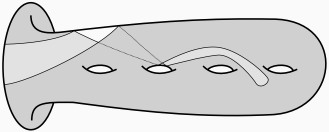

So can try to get at $BDiff_\partial(D^d)$ by understanding the other two terms.

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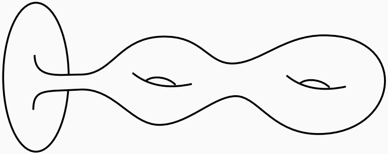
For example, when $d = 2n$ can take

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$



for “arbitrarily large” g.

$$BDiff_\partial(D^{2n}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BEmb^{\cong}_{1/2\partial}(W_{g,1})$$

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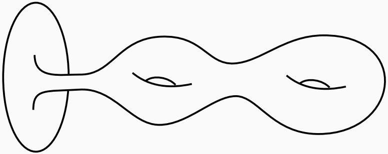

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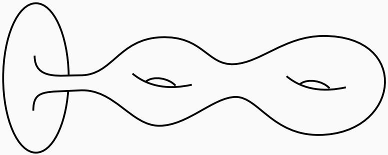

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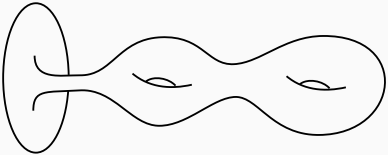

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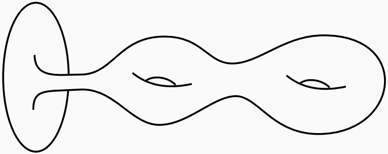

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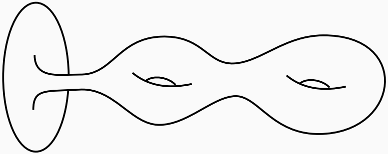

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The two results deal with these difficulties in very different ways.

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The proposal is to consider all $BTop(\mathbb{R}^d)$ at once, as the functor

$$\begin{aligned} \mathrm{Bt} : \left\{ \begin{matrix} \text{category of finite-dimensional} \\ \text{inner product spaces} \end{matrix} \right\} &\longrightarrow \left\{ \begin{matrix} \text{category of based} \\ \text{topological spaces} \end{matrix} \right\} \\ V &\longmapsto BTop(V) \end{aligned}$$

Orthogonal calculus

Orthogonal calculus considers continuous functors

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Any functor F has a best approximation $\mathsf{F} \to T_r\mathsf{F}$ by a polynomial functor of degree $\leq r$, assembling to a "Taylor tower".

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & \nearrow & T_2\mathsf{F} \\ & & \downarrow \\ & \nearrow & T_1\mathsf{F} \\ & & \downarrow \\ \mathsf{F} & \longrightarrow & T_0\mathsf{F} \end{array}$$

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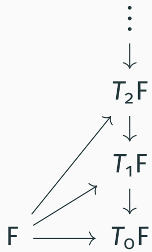

Homogeneous polynomials $\text{hofib}(T_r F \to T_{r-1} F)$ have a very particular structure: they are

$$V \longmapsto \Omega^\infty(\Theta F^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$$

for an $O(r)$-spectrum $\Theta F^{(r)}$, the rth *derivative*.

Orthogonal calculus for $V \mapsto \mathrm{Bt}(V) = BTop(V)$

(0) $T_0 \mathrm{Bt}(V) = \operatorname{colim}_{n \to \infty} BTop(V \oplus \mathbb{R}^n) \simeq BTop$

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The "band" pattern in $\pi_*(BDiff_\partial(D^{2n}))_{\mathbb{Q}}$ suggests that this is the case for all the higher derivatives too.

How to describe them?

Higher derivatives

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This would be a remarkable relationship between homeomorphisms of Euclidean space, algebraic K- and L-theory, cyclic homology, and graph cohomology.