Tautological rings for smooth manifolds

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Plan

1. Describe some characteristic classes for smooth fibre bundles: Miller–Morita–Mumford classes.
2. Do an example calculation.
3. Form the ring of all Miller–Morita–Mumford classes for smooth bundles with fibre M: the tautological ring of M.
4. Explain some things that we know—and do not know—about these tautological rings.
5. More examples.

Reminder: Characteristic classes of vector bundles

Recall that a real, oriented, d-dimensional vector bundle $V \to X$ has *Pontrjagin classes*

$$p_i(V) \in H^{4i}(X)$$

and, if d is even, an *Euler class*

$$e(V) \in H^d(X).$$

I will always consider $\mathbb{Q}$-coefficients.

These provide a complete set of $\mathbb{Q}$-cohomological invariants of oriented vector bundles: The classifying space $BSO(d)$ for real, oriented, d-dimensional vector bundles has

$$H^*(BSO(d)) = \begin{cases} \mathbb{Q}[p_1, \ldots, p_{n-1}, e] & \text{if } d = 2n \\ \mathbb{Q}[p_1, \ldots, p_{n-1}, p_n] & \text{if } d = 2n+1. \end{cases}$$

In both cases $p_i = 0$ for $i > n$, and in the first case $p_n = e^2$.

Smooth fibre bundles

Let M be a compact closed smooth d-dimensional manifold.

A map $\pi : E \to B$ is a *smooth* fibre bundle with fibre M when it equipped with an open cover $\{U_\alpha\}_{\alpha \in I}$ of B and homeomorphisms

$$\varphi_\alpha : E|_{U_\alpha} := \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times M,$$

such that each

$$\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times M \longrightarrow (U_\alpha \cap U_\beta) \times M$$

is a continuous family of diffeomorphisms of M.

In this case the tangent spaces at each point on each fibre $\pi^{-1}(b)$ assemble to a vector bundle

$$T_\pi E \longrightarrow E,$$

the *vertical tangent bundle*.

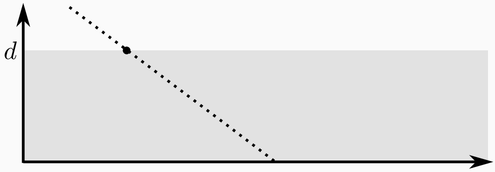
An *orientation* of the fibre bundle π is an orientation of the vector bundle $T_\pi E$, i.e. a continuous choice of orientation of the $\pi^{-1}(b)$.

Fibre integration

If $\pi : E \to B$ is an oriented smooth fibre bundle with fibre M, then its Serre spectral sequence provides a map

$$\int_\pi : H^{i+d}(E) \twoheadrightarrow E_\infty^{i,d} \subset E_2^{i,d} = H^i(B; b \mapsto H^d(\pi^{-1}(b))) = H^i(B)$$

called *integration along the fibre*, or the *Gysin map*.



In de Rham cohomology it is given by literally integrating $(i+d)$-forms on E along the d-dimensional fibres of π, to obtain i-forms on B.

Miller–Morita–Mumford classes

Let $\pi : E \to B$ be an oriented smooth fibre bundle with d-dimensional fibre, and c be a characteristic class of oriented d-dimensional vector bundles.

Then

$$\kappa_c(\pi) := \int_\pi c(T_\pi E) \in H^{|c|-d}(B),$$

is the *Miller–Morita–Mumford (MMM) class* associated to c.

If $|c| = d$ then $\kappa_c(\pi) \in H^0(B) = \mathbb{Q}$ is just a *characteristic number* of M. Thus the κ_c can be thought of as a generalisation of characteristic numbers from a single manifold to families of manifolds.

By construction they are natural with respect to pull-back of fibre bundles. In particular, if $|c| > d$ then they vanish for trivial bundles.

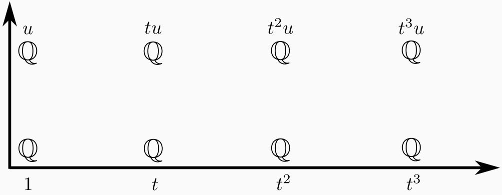
An example

Consider the standard action of $U(1)$ on $S^2 = \mathbb{C}^+$, and form the Borel construction

$$\pi : E := EU(1) \times_{U(1)} S^2 \longrightarrow BU(1) = \mathbb{CP}^\infty.$$

This is a smooth oriented fibre bundle with fibre S^2.

The Serre spectral sequence has the form



so collapses.

There is a multiplicative extension: $u^2 = -tu$.

An example (continued)

For oriented 2-dimensional vector bundles the only characteristic class is the Euler class e (and its powers).

One can show that $e(T_\pi E) = 2u + t$. Using $u^2 = -tu$ we have

$$e(T_\pi E)^2 = (2u + t)^2 = 4u^2 + 4tu + t^2 = t^2$$

and so

$$\begin{aligned} e(T_\pi E)^{2i} &= t^{2i} \\ e(T_\pi E)^{2i+1} &= (2u + t)t^{2i} = 2ut^{2i} + t^{2i+1}. \end{aligned}$$

Fibre integration extracts the coefficient of u, giving

$$\begin{aligned} \kappa_{e^{2i}}(\pi) &= 0 \\ \kappa_{e^{2i+1}}(\pi) &= 2t^{2i}. \end{aligned}$$

Thus MMM classes can be nontrivial!

Tautological rings

If $\pi : E \to B$ is an oriented smooth fibre bundle with d-dimensional fibre M, $\kappa_c \mapsto \kappa_c(\pi)$ gives a ring homomorphism

$$\Psi_\pi : \mathbb{Q}[\kappa_c : \substack{c \text{ a monomial in Euler and} \\ \text{Pontrjagin classes of degree } > d}] \longrightarrow H^*(B).$$

The *tautological ring* of M is

$$R^*(M) := \frac{\mathbb{Q}[\kappa_c : \substack{c \text{ a monomial in Euler and} \\ \text{Pontrjagin classes of degree } > d}]}{\bigcap_\pi \mathrm{Ker}(\Psi_\pi)},$$

where the intersection is taken over *all* smooth oriented fibre bundles π with fibre M.

That is, $\bigcap_\pi \mathrm{Ker}(\Psi_\pi)$ is the ideal of those relations among MMM classes which hold for *all* smooth fibre bundles with fibre M.

Goal: Study these rings, as invariants of M.

The case $d = 2$

When $M = \Sigma_g$ is an oriented surfaces of genus g, these rings have been studied in depth by algebraic geometers investigating the moduli space of genus g Riemann surfaces.

In this case $H^*(BSO(2)) = \mathbb{Q}[e]$ so, setting $\kappa_i := \kappa_{e^{i+1}}$, the tautological ring has the form

$$R^*(\Sigma_g) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]/I_g.$$

Faber has calculated

$$R^*(\Sigma_3) = \mathbb{Q}[\kappa_1]/(\kappa_1^2)$$
$$R^*(\Sigma_4) = \mathbb{Q}[\kappa_1]/(\kappa_1^3) \text{ with } \kappa_2 = \tfrac{3}{32}\kappa_1^2$$
$$R^*(\Sigma_5) = \mathbb{Q}[\kappa_1]/(\kappa_1^4) \text{ with } \kappa_2 = \tfrac{5}{72}\kappa_1^2, \kappa_3 = \tfrac{1}{288}\kappa_1^3$$
$$R^*(\Sigma_6) = \mathbb{Q}[\kappa_1, \kappa_2]/(127\kappa_1^3 - 2304\kappa_1\kappa_2, 113\kappa_1^4 - 36864\kappa_2^2)$$
$$\text{with } \kappa_3 = \tfrac{5}{2304}\kappa_1^3, \kappa_4 = \tfrac{5}{73728}\kappa_1^4$$

(and has computed $R^*(\Sigma_g)$ for $g < 24$).

Higher dimensions

There is a high-dimensional analogue of the genus g surface: the $2n$-dimensional manifold

$$W_g := \#^g S^n \times S^n.$$

Theorem [Grigoriev]. If n is odd and $g > 1$ then the $\mathbb{Q}$-algebra $R^*(W_g)$ is finitely-generated.

Theorem [Galatius–Grigoriev–R-W]. If n is odd then

(i) $R^*(W_0)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_n}]$,

(ii) $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,

(iii) $R^*(W_g)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \ldots, \kappa_{ep_{n-1}}]$ for all $g > 1$.

Finite generation

Inspired by Grigoriev's result, I investigated the finite-generation properties of $R^*(M)$ for more general M:

Theorem [R-W]. If M is an oriented closed $2n$-manifold and either

(H1) $H^*(M;\mathbb{Q})$ is non-zero only in even degrees, or

(H2) $H^*(M;\mathbb{Q})$ is non-zero only in degrees 0, $2n$, and odd degrees, and $\chi(M) \neq 0$,

then $R^*(M)$ is finitely-generated. e.g. W_g^{2n} except $(n, g) = (\text{odd}, 1)$.

Theorem [R-W]. For $k \geq 5$ odd there are simply-connected manifolds M having the integral homology of

$$(S^2 \times S^{2k+2})\#(S^2 \times S^{2k+2})\#(S^3 \times S^{2k+1})\#(S^k \times S^{k+4})\#(S^k \times S^{k+4})$$

for which $R^*(M)$ is *not* finitely-generated.

Methods

To investigate $R^*(M)$ there are two things which need to be done:

(1) Get upper bounds on $R^*(M)$, by finding relations among MMM-classes which hold for *all* fibre bundles with fibre M.

Grigoriev did this by carefully studying the multiplicative structure of the Serre spectral sequence; my generalisation of his results instead uses parameterised stable homotopy theory and trace identities.

In both cases, the method does not really use the smooth structure: the relations given by these methods hold also for e.g. topological fibre bundles (though they can then be manipulated in ways that are specific to smooth bundles).

This is not really satisfactory: surely the smooth structure should impose further relations?

Methods (continued)

(2) Get lower bounds on $R^*(M)$ by finding bundles for which MMM-classes do not vanish.

The only systematic method I know for this is to generalise the Example: look for k-torus actions $T \curvearrowright M$, form the Borel construction $\pi : ET \times_T M \to BT$, and study the image of

$$\Psi_\pi : R^*(M) \longrightarrow H^*(BT) = \mathbb{Q}[t_1, t_2, \ldots, t_k]$$

in terms of fixed-point data, using the localisation theorem in equivariant cohomology.

Also not very satisfactory: there are certainly bundles not coming from compact Lie group actions. For example, this map kills nilpotent elements so can at best only tell us about $R^*(M)/\sqrt{0}$.

Surely there are new methods to be tried in both directions.

More examples

In dimension 4 we have $H^*(BSO(4)) = \mathbb{Q}[p_1, e]$.

Using every technique I know, I was able to show that $R^*(\mathbb{CP}^2)$ is generated by $\kappa_{p_1^2}$, $\kappa_{p_1^4}$, and κ_{ep_1}, that the relations

$$(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})\kappa_{p_1^4}$$
$$(4\kappa_{p_1^2} - 7\kappa_{ep_1})(\kappa_{p_1^2} - 2\kappa_{ep_1})(21\kappa_{ep_1} + 8\kappa_{p_1^2})$$
$$(4\kappa_{p_1^2} - 7\kappa_{ep_1})(316\kappa_{ep_1}^3 - 343\kappa_{p_1^4})$$
$$(4\kappa_{p_1^2} - 7\kappa_{ep_1})(1264\kappa_{p_1^2}\kappa_{ep_1}^2 + 2212\kappa_{ep_1}^3 - 5145\kappa_{p_1^4}).$$

hold, and that there is a ring epimorphism

$$R^*(\mathbb{CP}^2) \longrightarrow \mathbb{Q}[\kappa_{p_1^2}, \kappa_{ep_1}, \kappa_{p_1^4}]/(7\kappa_{ep_1} - 4\kappa_{p_1^2}).$$

Using *family Seiberg–Witten theory*, Baraglia has shown that in fact the relation $7\kappa_{ep_1} - 4\kappa_{p_1^2}$ holds, so

$$R^*(\mathbb{CP}^2) \cong \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^4}].$$

He also showed that $R^*(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong \mathbb{Q}[\kappa_{p_1^2}, \kappa_{p_1^3}]$

More examples (continued)

Using every technique I know, I was able to show that $R^*(S^2 \times S^2)$ is generated by $\kappa_{p_1^2}$, $\kappa_{p_1^3}$, κ_{ep_1}, $\kappa_{ep_1^2}$, κ_{e^3}, $\kappa_{e^3p_1}$, and κ_{e^5}. I was also able to find a very large ideal of relations among these 7 variables, and this ideal has codimension 3, so

$$\text{Krull dimension of } R^*(S^2 \times S^2) \leq 4.$$

On the other hand, using countably-many different 2-torus actions on $S^2 \times S^2$ I was able to give an epimorphism

$$R^*(S^2 \times S^2) \longrightarrow \frac{\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_1^2}, \kappa_{e^3}, \kappa_{e^3p_1}]}{\left(\begin{matrix}\kappa^3_{ep_1}\kappa_{e^3p_1} - \kappa^2_{ep_1}\kappa_{ep_1^2}\kappa_{e^3} + \kappa^2_{ep_1}\kappa^2_{e^3} - 4\kappa_{ep_1}\kappa_{ep_1^2}\kappa_{e^3p_1} \\ -8\kappa_{ep_1}\kappa_{e^3}\kappa_{e^3p_1} + 4\kappa^2_{ep_1^2}\kappa_{e^3} + 16\kappa^2_{e^3p_1}\end{matrix}\right)}$$

so $R^*(S^2 \times S^2)$ has Krull dimension ≥ 3.

Question: is it 3 or 4? **New ideas are required!**

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