Cohomology of moduli spaces

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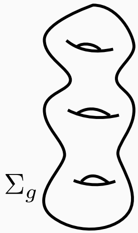
Which moduli spaces?

Moduli spaces of Riemann surfaces



For $g \geq 2$,

$$\mathcal{M}_g = \frac{\{\text{genus } g \text{ Riemann surfaces}\}}{\text{isomorphism}}$$

Earle–Eells '69:

$$\begin{aligned}\mathcal{M}_g &= \{\text{complex structures on } \Sigma_g\} /\!\!/ \mathit{Diff}^+(\Sigma_g) \\ &= \frac{\{\text{complex structures on } \Sigma_g\}}{\mathit{Diff}^+(\Sigma_g)_{id}} /\!\!/ \frac{\mathit{Diff}^+(\Sigma_g)}{\mathit{Diff}^+(\Sigma_g)_{id}} \\ &=: (\text{Teichmüller space}) /\!\!/ (\text{mapping class group } \Gamma_g)\end{aligned}$$

$\{$complex structures on $\Sigma_g\}$ and Teichmüller space are contractible

Moduli spaces of Riemann surfaces

Reminder on classifying spaces
G (topological) group

$BG = \{\text{any contractible free } G\text{-space}\}/G$ "classifying space of G"

G discrete

$BG = K(G, 1)$ Eilenberg–Mac Lane space, recover G as $\pi_1(BG, *)$
$H_*(BG) = H_*^{group}(G)$

G topological

recover G up to homotopy as $\Omega BG = map_*(S^1, BG)$

combinatorial group theory

$$\mathcal{M}_g \simeq B\Gamma_g \simeq BDiff^+(\Sigma_g)$$

algebraic geometry differential topology

Examples of moduli spaces

Anything which is like $\mathcal{M}_g$, $B\Gamma_g$, or $BDiff^+(\Sigma_g)$ is a moduli space.

Like $\mathcal{M}_g$:

- $\mathcal{A}_g$ moduli spaces of principally polarized abelian varieties
- $Conf_n(\mathbb{R}^k)$ unordered configuration spaces
- Hur_n^G Hurwitz spaces

Like $B\Gamma_g$:

- BS_n symmetric groups
- $B\beta_n$ braid groups
- $BGL_n(R)$ general linear groups
- $BSp_{2n}(R)$ symplectic groups
- $BAut(F_n)$ automorphism groups of free groups

Like $BDiff^+(\Sigma_g)$:

- $BDiff(W)$ diffeomorphism groups
- $BhAut(W)$ homotopy automorphism groups

Moduli spaces of smooth manifolds

For W a closed manifold, the space of embeddings

$$Emb(W, \mathbb{R}^\infty)$$

has a free $Diff(W)$-action by precomposition, and is contractible.

Take the specific geometric model

$$\begin{aligned} BDiff(W) &= Emb(W, \mathbb{R}^\infty)/Diff(W) \\ &= \left\{ X \subset \mathbb{R}^\infty \mid \substack{X \text{ is a smooth submanifold} \\ \text{which is diffeomorphic to } W} \right\} \end{aligned}$$

the "moduli space of submanifolds of $\mathbb{R}^\infty$ diffeomorphic to W".

Why cohomology?

A moduli space $\mathcal{M}$ is anything which classifies some kind of families:

$$map(B, \mathcal{M}) = \{\text{families of (...) over } B\}$$

Then

$$H^*(\mathcal{M}) = \{\text{characteristic classes of such families}\}$$

Homological stability

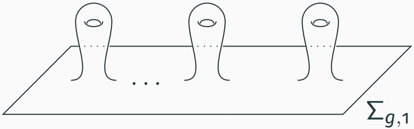
Stabilisation

$S_{n-1} \to S_n$: $\quad \sigma \longmapsto \sigma \sqcup Id_n$

$GL_{n-1}(R) \to GL_n(R)$: $\quad A \longmapsto A \oplus Id_R = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$

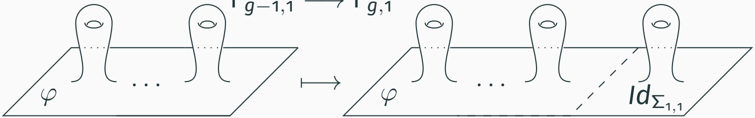
$Aut(F_{n-1}) \to Aut(F_n)$: $\quad f \longmapsto f * Id_{\mathbb{Z}}$

For mapping class groups form the variant $\Gamma_{g,1} = \frac{Diff(\Sigma_{g,1})}{Diff(\Sigma_{g,1})_{id}}$, where $Diff(\Sigma_{g,1})$ is diffeomorphisms which are the identity near the boundary.



Then boundary connect-sum with $\Sigma_{1,1}$ gives

$\Gamma_{g-1,1} \longrightarrow \Gamma_{g,1}$



Homological stability

Many sequences of groups $G_0 \to G_1 \to G_2 \to G_3 \to \cdots$ satisfy

Generic homological stability theorem. For a divergent function f

$$H_d(G_{n-1}) \to H_d(G_n) \text{ is an } \begin{cases} \text{epimorphism for } d \leq f(n), \\ \text{isomorphism for } d < f(n). \end{cases}$$

Equivalently, $H_d(G_n, G_{n-1}) = 0$ for $d \leq f(n)$.

- S_n (Nakaoka '60)
- $GL_n(R)$ (Quillen, Maazen '79, Charney '80, van der Kallen '80, ...)
- $Sp_{2n}(R)$, $O_{n,n}(R)$ (Vogtmann '81, Charney '87, ...)
- $Aut(F_n)$ (Hatcher–Vogtmann '98, ...)
- $\Gamma_{g,1}$ (Harer '85, Ivanov '91, Boldsen '12, R-W '16)

$$\text{specifically } H_d(\Gamma_{g,1}, \Gamma_{g-1,1}) = 0 \text{ for } d \leq \frac{2g-2}{3}.$$

Closing the boundary $\Gamma_{g,1} \to \Gamma_g$ is also a homology isomorphism in a stable range, so $H_d(\mathcal{M}_g) \cong H_d(\Gamma_{g,1})$ stably.

Proof overview

Idea of Quillen: find simplicial object

$$X_n(0) \leftleftarrows X_n(1) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; X_n(2) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; \cdots$$

with G_n-action, such that

(i) stabilisers of simplices are smaller groups in the family,
(ii) X_n is a good approximation to a point: it is highly connected,
(iii) X_n/G_n is not too complicated.

By (ii) $X_n /\!\!/ G_n$ is a good approximation to BG_n; by (i) it is constructed from BG_i with $i < n$, and by (iii) the recipe for this construction is not too complicated.

Axiomatised (Wahl–R-W '17), but verifying (ii)—which was always the most difficult—must still be done by hand.

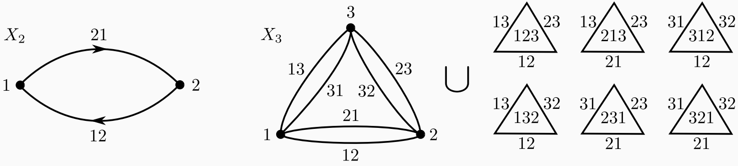
It always depends on the specifics of the groups G_n, though there are some general principles.

Example: symmetric groups

For S_n use the *complex of injective words*:

$$X_n(p) := \left\{ \begin{matrix} \text{words } a_0 a_1 \cdots a_p \text{ in } \{1, 2, \ldots, n\} \text{ where} \\ \text{each letter occurs at most once} \end{matrix} \right\},$$

where the ith face of a simplex is given by removing the ith letter.



Theorem (Farmer '79, Björner–Wachs '83, Kerz '04, R-W '13, Bestvina '14, Gan '17)
X_n is $(n-2)$-connected.

Strategy can be reinterpreted as a "partial nonabelian resolution"

$$BG_n \longleftarrow X_n(0)/\!\!/G_n \;\overleftarrow{\overleftarrow{\quad}}\; X_n(1)/\!\!/G_n \;\overleftarrow{\overleftarrow{\overleftarrow{\quad}}}\; X_n(2)/\!\!/G_n \;\overleftarrow{\overleftarrow{\overleftarrow{\overleftarrow{\quad}}}}\; \cdots$$

where each $X_n(p)/\!\!/G_n$ is a coproduct of BG_i's with $i < n$.

This "resolution" point of view applies to many further examples, such as $Conf_n(\mathbb{R}^k)$ or $BDiff(W)$.

Theorem (Galatius–R-W '18)
For $2n \geq 6$ and W a simply-connected $2n$-manifold with $\partial W \neq \varnothing$ have

$$H_d(BDiff(W\#^g S^n \times S^n), BDiff(W\#^{g-1} S^n \times S^n)) = 0 \text{ for } d \leq \frac{g-2}{2}.$$

Variants: virtually poly-$\mathbb{Z}$ fundamental group (Friedrich), for stabilisation by $S^p \times S^q$ etc. (Perlmutter), for homeomorphism groups (Kupers), for *Diff* as a discrete group (Nariman).

Stable cohomology

Stable (co)homology

Complementary to the stability problem is the determination of the limiting homology.

These two problems are unrelated at the level of techniques, and also logically.

Philosophy for approaching the limiting homology comes from the foundations of algebraic K-theory.

Group-completion

Combine moduli spaces into a monoid:

$\coprod_{n\geq 0} BS_n$	disjoint union of finite sets	$\coprod_{n\geq 0} BGL_n(R)$	direct sum of R-modules
$\coprod_{n\geq 0} Conf_n(\mathbb{R}^k)$	disjoint union of configurations	$\coprod_{g\geq 0} B\Gamma_{g,1}$	boundary connect-sum of surfaces

Recall for a topological group G have $G \simeq \Omega BG$.

Can also form classifying space BM of a topological monoid M, and ΩBM is its *homotopy theoretic group-completion*.

How does this relate to "inverting elements of M"?

Group-completion theorem.
If M is a homotopy commutative monoid then

$$H_*(M)[\pi_0(M)^{-1}] = H_*(\Omega BM).$$

e.g. $M = \coprod_{n\geq 0} X_n$ *with* X_n *connected then* $\operatorname{colim}_{n\to\infty} H_*(X_n) = H_*(\Omega_0 BM)$.

Fundamentally a *homological* result.

Theorem (Barratt–Priddy '72, Quillen, Segal '73, May '72)

$$\Omega B\left(\coprod_{n\geq 0} Conf_n(\mathbb{R}^k)\right) \simeq \Omega^k S^k = map_*(S^k, S^k)$$

$$\Omega B\left(\coprod_{n\geq 0} BS_n\right) \simeq \Omega^\infty \mathbf{S} = \operatorname*{colim}_{k\to\infty} \Omega^k S^k$$

(Similarity because $BS_n = Emb(\{1,2,\ldots,n\},\mathbb{R}^\infty)/S_n = Conf_n(\mathbb{R}^\infty)$)

Cannot generally expect to identify a group-completion:

Definition (Quillen '72)

$$\Omega B\left(\coprod_{n\geq 0} BGL_n(R)\right) =: K(R), \text{ algebraic } K\text{-theory}$$

Theorem (Madsen–Weiss ’07)

$$\Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,1}\right) \simeq \Omega^\infty \mathbf{MTSO}(2)$$

Easy to calculate with $\mathbb{Q}$-coefficients:

$$\lim_{g\to\infty} H^*(\Gamma_{g,1};\mathbb{Q}) = H^*(\Omega^\infty_0\mathbf{MTSO}(2);\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\ldots], \quad |\kappa_i| = 2i$$

(Also known with $\mathbb{F}_p$-coefficients (Galatius ’04), more complicated.)

$\Rightarrow$ Mumford’s ’83 conjecture that $H^*(\mathcal{M}_g;\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\ldots]$ stably.

The Madsen–Weiss theorem

There are now several further proofs of the Madsen–Weiss theorem

- Galatius–Madsen–Tillmann–Weiss '09
- Galatius–R-W '10
- Eliashberg–Galatius–Mishachev '11

All proofs are from the point of view of differential topology, and begin with the replacement $B\Gamma_{g,1} \simeq BDiff(\Sigma_{g,1})$ and the specific geometric model

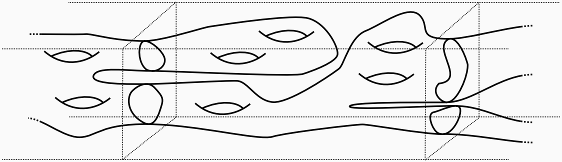
$$BDiff(W) = \{X \subset \mathbb{R}^\infty \mid {}^{X \text{ is a smooth submanifold}}_{\text{which is diffeomorphic to } W}\}$$

and its analogue for manifolds with boundary.

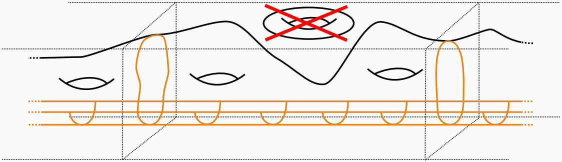
The most well-developed method of proof (GMTW and GR-W) builds on the remarkable theorem of Tillmann '97 relating $\Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,1}\right)$ with the cobordism categories from TCFT.

Proof overview (GR-W)

- Consider space of "long surfaces": surfaces inside $\mathbb{R} \times [0,1]^\infty$ which can move to $\pm\infty$. This is a model for $BCob_2$.



- (GMTW theorem) Show this is $\Omega^{\infty-1}\mathbf{MTSO}(2)$, "$h$-principle".
- Geometric model for $B\left(\coprod_{g\geq 0} B\Gamma_{g,1}\right)$ as subspace of special long surfaces.



- Show that an arbitrary family of long surfaces can be deformed into a family of special ones, "parameterised surgery".

Further examples

The argument scheme just described was introduced by Galatius to study the stable homology of moduli spaces of graphs in $\mathbb{R}^\infty$, which model $BAut(F_n)$.

Theorem (Galatius '11)

$$\lim_{n\to\infty} H^*(Aut(F_n)) = H^*(\Omega_0^\infty \mathbf{S})$$

Corollary
$S_n \to Aut(F_n)$ is a (co)homology isomorphism in a stable range

Further examples

Galatius–R-W: analogue of the Madsen–Weiss theorem for *any manifold of dimension* $2n$ with respect to stabilisation by $S^n \times S^n$, i.e. a homotopy theoretic formula for

$$\lim_{g\to\infty} H^*(BDiff(W\#^g S^n \times S^n)).$$

The general formulation is quite complicated.

Special case: For $W = D^{2n}$ have

$$\begin{aligned}\lim_{g\to\infty} H^*(BDiff(D^{2n}\#^g S^n \times S^n)) &= H^*(\Omega_0^\infty \mathbf{MT}\theta_\mathbf{n}) \\ &\cong_\mathbb{Q} \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]\end{aligned}$$

where $\mathcal{B}$ is the basis of monomials of $\mathbb{Q}[e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}]$ of degree $> 2n$ (where $|e| = 2n$, $|p_i| = 4i$).

Beyond stable homology

Secondary homological stability

Homological stability for mapping class groups said

$$H_d(\Gamma_{g,1}, \Gamma_{g-1,1}) = 0 \text{ for } d \leq \frac{2g-2}{3}.$$

Theorem (Galatius–Kupers–R-W '19)
There are maps

$$\varphi_* \colon H_{d-2}(\Gamma_{g-3,1}, \Gamma_{g-4,1}) \longrightarrow H_d(\Gamma_{g,1}, \Gamma_{g-1,1})$$

which are epimorphisms for $d \leq \frac{3g-1}{4}$ *and isomorphisms for* $d \leq \frac{3g-5}{4}$.

"The failure of homological stability is itself stable."

Furthermore, with $\mathbb{Q}$*-coefficients these maps are epimorphisms for* $d \leq \frac{4g-1}{5}$ *and isomorphisms for* $d \leq \frac{4g-6}{5}$.

Chart

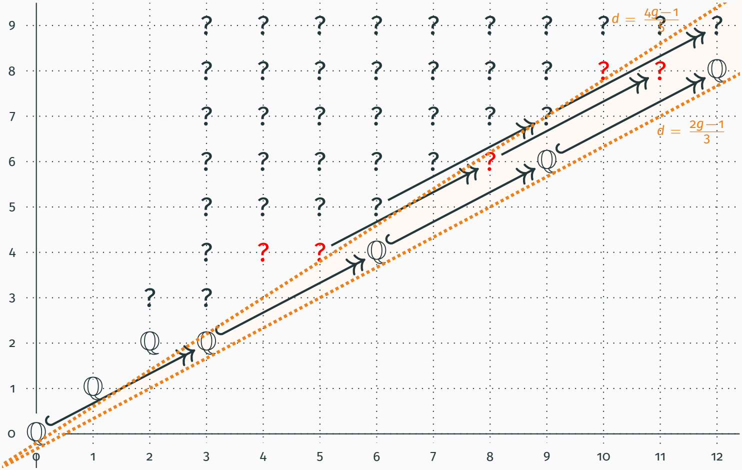


Figure 1: $H_d(\Gamma_{g,1}, \Gamma_{g-1,1}; \mathbb{Q})$; ? means unknown, ? means not zero

Non-zero groups: use results of Faber, Kontsevich, Morita.

Proof overview

Uses a new paradigm for understanding families of moduli spaces

S. Galatius, A. Kupers, O. R-W, *Cellular E_k-algebras* (arXiv:1805.07184)

based on methods from abstract homotopy theory.

Take seriously the idea that things such as

$$\mathbf{R} := \coprod_{g \geq 0} B\Gamma_{g,1}$$

are algebraic objects, and try to understand a "presentation".

Homotopical, so "higher algebraic": E_k-algebras.
e.g. $\mathbf{R}$ is an $\mathbb{N}$-graded E_2-algebra.

(Even *defining* the secondary stabilisation map requires this.)

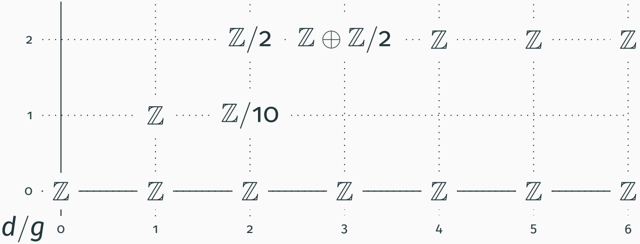
In homotopy theory

"generators" and "relations" $=$ cells

so: analyse cell structure of $\mathbf{R}$ as a $\mathbb{N}$-graded E_2-algebra.

Proof overview

Step 1. Reverse engineer E_2-cell structure of $\mathbf{R}$ in low degrees from calculations of $H_{g,d}(\mathbf{R}) = H_d(\Gamma_{g,1})$, giving a skeleton $\mathbf{C}$.



Many calculations available: Abhau, Benson, Bödigheimer, Boes, F. Cohen, Ehrenfried, Godin, Harer, Hermann, Korkmaz, Looijenga, Meyer, Morita, Mumford, Pitsch, Sakasai, Stipsicz, Tommasi, Wang, ...

The skeleton $\mathbf{C}$ has 5 cells. Can formulate (secondary) homological stability for $\mathbf{C}$, and prove it essentially by direct calculation.

Proof overview

Step 2. Show $\mathbf{R} = \mathbf{C} \cup \{E_2\text{-}(g, d)\text{-cells with } d \geq g - 1 \text{ and } d \geq 3\}$.

Lowest slope cell added to **C** is then a $(4, 3)$-cell; easy to see that this cannot break (secondary) stability in degrees $d < \frac{3g}{4}$.

To estimate cells, use a homology theory for E_k-algebras,

$$\begin{aligned} \text{“}E_k\text{-homology”} &= \text{“derived indecomposables”} \\ &= \text{“topological Quillen homology”} \end{aligned}$$

which detects E_k-cell structures.

Theorem (Galatius–Kupers–R-W ’19)
$H^{E_2}_{g,d}(\mathbf{R}) = 0$ *for* $d < g - 1$.

Comes down to proving connectivity of certain simplicial complexes of arcs on surfaces, but different to those in classical proofs of homological stability.

Further examples

This E_k-cellular perspective can be applied to many examples.

The vanishing line for E_k-homology (Step 2) holds for

- $\coprod_{n\geq 0} BGL_n(R)$ for R a field or Dedekind domain (Charney)
- $\coprod_{n\geq 0} BSp_{2n}(\mathbb{Z})$ (Looijenga–van der Kallen)
- $\coprod_{n\geq 0} BAut(F_n)$ (Hatcher–Vogtmann)

By itself such a vanishing line implies ordinary homological stability (with slope $\frac{1}{2}$).

To get more out one must investigate the E_k-cell structure in low degrees: phenomena here will propagate. For example

Theorem (Galatius–Kupers–R-W '19)
If $q = p^r \neq 2$, then $H_d(GL_n(\mathbb{F}_q); \mathbb{F}_p) = 0$ for $0 < d < n + r(p-1) - 2$.

If $q = 2$, then $H_d(GL_n(\mathbb{F}_2); \mathbb{F}_2) = 0$ for $0 < d < \frac{2n}{3} - 1$.

Questions?