Homeomorphisms of Euclidean space

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My goal is to explain some recent advances in understanding the algebraic topology of the topological group

$$Top(d) = Homeo(\mathbb{R}^d)$$

of homeomorphisms of $\mathbb{R}^d$ (with the usual compact-open topology).

I will try to express what we know via *homotopy groups*

$$\pi_n(X) = \frac{\{\text{continuous based maps } f : S^n \to X\}}{\text{homotopy}}$$

or, suppressing torsion, their rationalisations $\pi_n(X) \otimes \mathbb{Q}$.

To study the algebraic topology of a topological group G *as a group*, it is usual to investigate it's so-called classifying space

$$BG := \{\text{some contractible free } G\text{-space}\}/G.$$

We have $\pi_n(BG) = \pi_{n-1}(G)$, so there is not much difference between considering G or BG from this point of view.

Warm-up: Diffeomorphisms

Diffeomorphisms of Euclidean space

Let me start with the simpler example of the group

$$\mathrm{Diff}(\mathbb{R}^d)$$

of diffeomorphisms of $\mathbb{R}^d$ (with the weak C^∞-topology).

Why is this simpler?

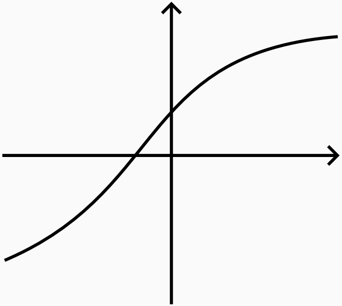
Linearising

$f : \mathbb{R}^d \to \mathbb{R}^d$ a diffeomorphism, consider

$$f_t(x) = \frac{f(t \cdot x) - f(0)}{t} + t \cdot f(0)$$

for $t \in [0, 1]$.

$$\Rightarrow \mathit{Diff}(\mathbb{R}^d) \simeq GL_d(\mathbb{R})$$



The Gram–Schmidt process deforms $GL_d(\mathbb{R})$ to its subgroup $O(d)$.

Diffeomorphisms of Euclidean space

The topology of $O(d)$ can be understood inductively, as $O(d+1)$ acts transitively on S^d with stabiliser $O(d)$: $\frac{O(d+1)}{O(d)} = S^d$.

This gives an exact sequence of homotopy groups

$$\cdots \to \pi_{i+1}(S^d) \to \pi_i(O(d)) \to \pi_i(O(d+1)) \to \pi_i(S^d) \to \pi_{i-1}(O(d)) \to \cdots$$

Much is known about the topology of spheres.

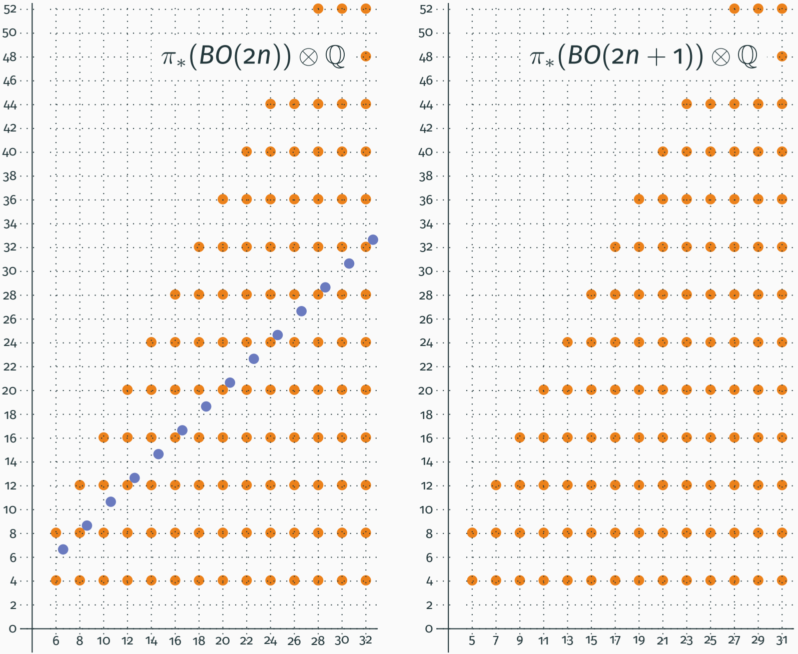
1. It is a fundamental (and easy) result in algebraic topology that $\pi_i(S^d) = 0$ for $i < d$.

$$\Rightarrow \pi_i(BO(d)) = \pi_i(BO(d+1)) \text{ for } i < d.$$

2. The calculation of $\pi_*(S^d) \otimes \mathbb{Q}$ is also fundamental (but less easy).

$$\Rightarrow \pi_*(BO(d)) \otimes \mathbb{Q} = \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ 0 & d \text{ odd}. \end{cases}$$

In particular in the limit $d \to \infty$ we have $\pi_*(BO) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i]$.



$\pi_*(BO(2n)) \otimes \mathbb{Q}$
$\pi_*(BO(2n+1)) \otimes \mathbb{Q}$
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52
6 8 10 12 14 16 18 20 22 24 26 28 30 32
5 7 9 11 13 15 17 19 21 23 25 27 29 31

The differences $\frac{O(d+1)}{O(d)}$

That the differences $\frac{O(d+1)}{O(d)} = S^d$ have a uniform description suggests that something is going on.

Indeed, these differences can be related to one another:

$$\frac{O(d+1)}{O(d)} \longrightarrow \Omega \frac{O(d+2)}{O(d+1)} := map_*(S^1, \frac{O(d+2)}{O(d+1)})$$
$$O(d) \cdot A \longmapsto (\theta \mapsto O(d+1) \cdot R_\theta (A \oplus 1) R_\theta^{-1}),$$

where $R_\theta \in O(d+2)$ rotates by θ in the last two coordinates.

The homotopy groups of the source S^d and the target ΩS^{d+1} of this map vanish in degrees $* < d$, but this map is an isomorphism on homotopy groups in degrees $* < 2d-1$: this is *Freudenthal's suspension theorem*.

$$\Rightarrow \pi_{d+i}(\tfrac{O(d+1)}{O(d)}) \cong \pi_{d+1+i}(\tfrac{O(d+2)}{O(d+1)}) \text{ for } i < d-1$$

This common value is the ith *stable homotopy group of the sphere spectrum* $\pi_i(\mathbb{S})$.

Back to homeomorphisms

Stabilising homeomorphisms

There are stabilisation maps

$$\begin{array}{ccccccccc} Top(d) & \longrightarrow & Top(d+1) & \longrightarrow & Top(d+2) & \longrightarrow & \cdots & \longrightarrow & Top \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\ O(d) & \longrightarrow & O(d+1) & \longrightarrow & O(d+2) & \longrightarrow & \cdots & \longrightarrow & O \end{array}$$

given by $- \times \mathbb{R}$ and *smoothing theory* identifies

$\pi_n(\frac{Top}{O}) \cong \Theta_n :=$ {smooth oriented n-manifolds homeomorphic to S^n},

the group of so-called *homotopy n-spheres* (not quite true for $n \leq 4$).

The theorem of Kervaire–Milnor '63 determines these groups:

$$\begin{array}{llll} \Theta_5 = 0 & \Theta_6 = 0 & \Theta_7 = \mathbb{Z}/28 & \Theta_8 = \mathbb{Z}/2 \\ \Theta_9 = (\mathbb{Z}/2)^3 & \Theta_{10} = \mathbb{Z}/6 & \Theta_{11} = \mathbb{Z}/992 & \Theta_{12} = 0 \end{array}$$

and in particular shows that they are all finite abelian groups.

$$\Rightarrow \pi_*(BTop) \otimes \mathbb{Q} \cong \pi_*(BO) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i]$$

The differences $\frac{Top(d+1)}{Top(d)}$

In distinction with $\frac{O(d+1)}{O(d)} = S^d$, the spaces $\frac{Top(d+1)}{Top(d)}$ cannot be identified with anything previously known: they are their own thing.

They still have vanishing homotopy groups in degrees $* < d$.

The same rotation construction as before gives maps

$$\frac{Top(d+1)}{Top(d)} \longrightarrow \Omega \frac{Top(d+2)}{Top(d+1)},$$

though now these are only known to give isomorphisms on homotopy groups in degrees $* \lesssim \frac{4}{3}d$ (Igusa '88).

Theorem (Waldhausen '81). In this range

$$\pi_{d+i}\left(\frac{Top(d+1)}{Top(d)}\right) = K_i(\mathbb{S}),$$

the ith "algebraic K-theory of the sphere spectrum".

(I don't expect you to know what this means.)

The differences $\frac{Top(d+1)}{Top(d)}$

When we work rationally the ring spectrum $\mathbb{S}$ can be replaced by the ordinary ring $\mathbb{Z}$:

$$K_*(\mathbb{S}) \otimes \mathbb{Q} = K_*(\mathbb{Z}) \otimes \mathbb{Q}$$

and combined with the calculation (Borel '74) of the latter gives

$$\pi_{d+*}\left(\frac{Top(d+1)}{Top(d)}\right) \otimes \mathbb{Q} = \mathbb{Q}[0] \oplus \mathbb{Q}[5] \oplus \mathbb{Q}[9] \oplus \mathbb{Q}[13] \oplus \mathbb{Q}[17] \oplus \cdots$$

for $d + * \lesssim \frac{4}{3}d$.

This leads to the formula (Farrell–Hsiang '78)

$$\pi_*(BTop(d)) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \begin{cases} \mathbb{Q}[d] & d \text{ even} \\ \bigoplus_{j \geq 1} \mathbb{Q}[d+1+4j] & d \text{ odd.} \end{cases}$$

in the Igusa stable range $* \lesssim \frac{4}{3}d$.

Patterns

A pattern for $Top(2n)$

The story so far was complete by 1988, and not much had changed until recently. The impetus has been a '15 theorem of Weiss on "topological Pontrjagin classes", and especially a perspective adopted in his argument.

Contemplating this perspective led Kupers and I to the following:

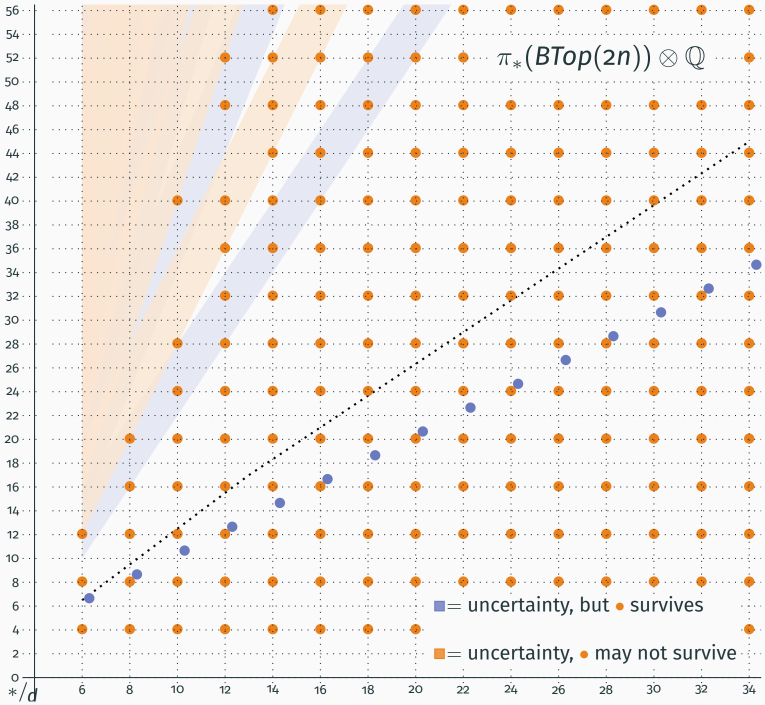
Theorem (Kupers–R-W '20). For $d = 2n \geq 6$ we have

$$\pi_*(BTop(2n)) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \mathbb{Q}[2n]$$

modulo classes in the bands of degrees

$$\bigcup_{s \geq 3} [2s(n-2)+4, 2s(n-1)+4].$$

These bands have slopes $3d, 4d, 5d, 6d, \ldots$



$\pi_*(BTop(2n)) \otimes \mathbb{Q}$
$*/d$
= uncertainty, but • survives
= uncertainty, • may not survive

Calculations for $Top(2n+1)$

Using different techniques, Krannich and I investigated $Top(2n+1)$ outside of the Igusa stable range.

Theorem (Krannich–R-W '21). For $d = 2n+1 \geq 5$ we have

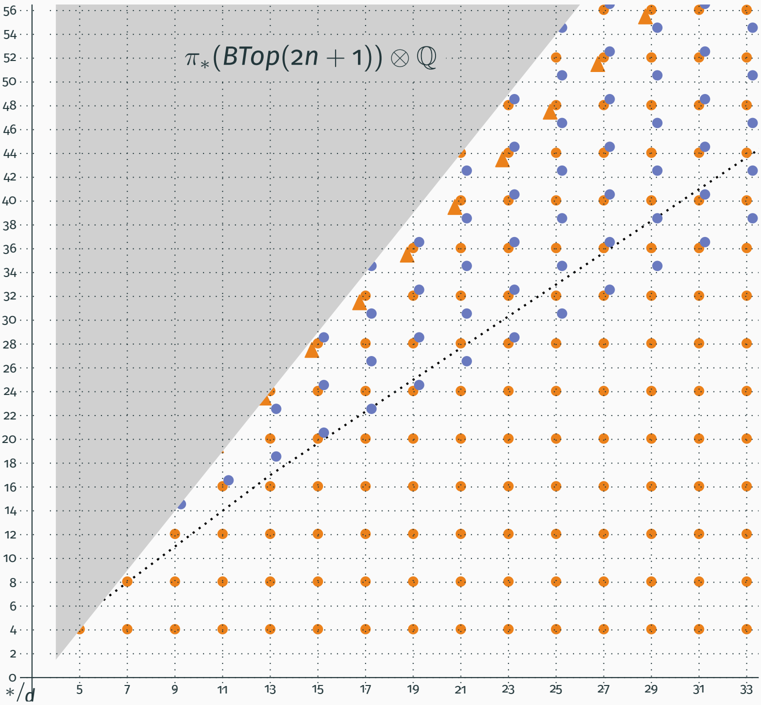
$$\pi_*(BTop(2n+1)) \otimes \mathbb{Q} = \bigoplus_{i \geq 1} \mathbb{Q}[4i] \oplus \bigoplus_{j \geq 1} \mathbb{Q}[2n+2+4j] \oplus \mathbb{Q}[4n]$$

in degrees $* \leq 5n - 6 \sim \frac{5}{2}d$.

That is,

(i) The elements found by Farrell–Hsiang persist well beyond the Igusa stable range $* \lesssim \frac{4}{3}d$,

(ii) but they do not account for everything: there is a new phenomenon occurring in degree $4n = 2d - 2$.



$\pi_*(BTop(2n+1)) \otimes \mathbb{Q}$
$*/d$

A conjectural explanation

Proposal

The "band" picture suggests that $\pi_*(BTop(d)) \otimes \mathbb{Q}$ is a superposition of various phenomena happening on different "wavelengths"

The kinds of phenomena that occur depend only on the parity of d, but the rth phenomenon contributes to degrees around $r \cdot d$

i.e. these phenomena get "spread out" as d increases

Taking this as "experimental data", there is a mechanism from homotopy theory that could explain it:

Orthogonal Calculus

This tells us to consider all the $BTop(d)$ at once, as the functor

$$V \mapsto BTop(V) : \left\{\begin{array}{c}\text{category of finite-dimensional}\\ \text{inner product spaces}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of based}\\ \text{topological spaces}\end{array}\right\}$$

Orthogonal calculus

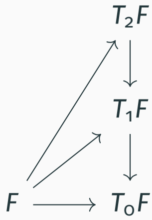
Weiss' orthogonal calculus proposes to consider such functors

$$F : \left\{\begin{smallmatrix}\text{category of finite-dimensional}\\ \text{inner product spaces}\end{smallmatrix}\right\} \longrightarrow \left\{\begin{smallmatrix}\text{category of based}\\ \text{topological spaces}\end{smallmatrix}\right\}$$

as though they were functions, and develop a notion of Taylor expansions for them.

There is a notion of derivative $F^{(1)}(V) := \text{fibre}(F(V) \to F(V \oplus \mathbb{R}))$ of such a functor, and hence of being polynomial of degree $\leq r$.

Any functor F has a best approximation $F \to T_r F$ by a polynomial functor of degree $\leq r$, assembling to a "Taylor tower".



One remarkable thing about this theory is that the homogeneous polynomials, i.e. the fibres of $T_r F \to T_{r-1} F$, have a very particular structure: they are

$$V \longmapsto \Omega^\infty(\Theta F^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$$

for some $O(r)$-spectrum $\Theta F^{(r)}$.

Orthogonal calculus for $V \mapsto BTop(V)$

Such a homogeneous functor $F(V) = \Omega^\infty(\Theta F^{(r)} \wedge_{O(r)} (\mathbb{R}^r \otimes V)^+)$ has precisely the behaviour we have observed

$$\begin{aligned}\pi_*(F(V \oplus \mathbb{R} \oplus \mathbb{R})) \otimes \mathbb{Q} &= \pi_{*-2r}(F(V)) \otimes \mathbb{Q} \\ &\neq \pi_{*-r}(F(V \oplus \mathbb{R})) \otimes \mathbb{Q} \text{ in general}\end{aligned}$$

The "band" pattern we have seen would then be explained by

(i) $BTop(V) \overset{\sim}{\to} T_\infty BTop(V)$ for $\dim(V)$ large enough
(ii) the known structure of $T_0 BTop(-) = BTop$ and $\Theta BTop^{(1)} = K(\mathbb{S})$
(iii) $\Theta BTop^{(r)} /\!\!/ SO(r)$ being a finite spectrum for each $r \geq 2$

The spectra $\Theta BTop^{(r)} /\!\!/ SO(r)$ would have to be very rich, with rational homotopy groups at least containing the r-loop part of Kontsevich's (even and odd) commutative graph cohomology, and most probably just being equal to this.

Krannich and I have identified $\Theta BTop^{(2)} \simeq_{\mathbb{Q}} \mathrm{coInd}_{O(1)}^{O(2)} \mathbb{S}^{-1}$, so the proposal looks good in this case.

Some ideas of the proofs

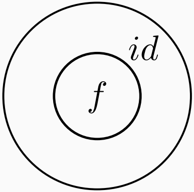
Smoothing theory

One cannot really study $Top(d)$ by thinking about homeomorphisms.

Instead, one uses "smoothing theory" in the manner of Morlet:

$$\frac{Homeo_\partial(D^d)}{Diff_\partial(D^d)} \simeq \Omega^d_0\left(\frac{Homeo(\mathbb{R}^d)}{Diff(\mathbb{R}^d)}\right) \simeq \Omega^d_0\left(\frac{Top(d)}{O(d)}\right).$$

Alexander trick: For $f : D^d \to D^d$ a homeomorphism fixing ∂D^d, consider



$$f_t(x) = \begin{cases} x & |x| \geq t \\ t \cdot f(x/t) & |x| \leq t. \end{cases}$$

$\Rightarrow Homeo_\partial(D^d) \simeq *$

$\Rightarrow BDiff_\partial(D^d) \simeq \Omega^d_0\left(\frac{Top(d)}{O(d)}\right)$

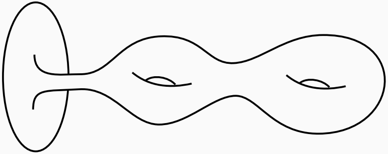
So understanding homeomorphisms of $\mathbb{R}^d$ is more or less the same as understanding diffeomorphisms of D^d, and this is how it is usually approached.

Stabilising by complexity

A programme of Galatius and myself, extending the Madsen–Weiss theorem to high dimensions, gives a good understanding of diffeomorphism groups of manifolds of dimension $2n$ which are "complicated" in the sense that they contain many $S^n \times S^n$'s.

In particular for the manifolds

$$W_{g,1} := D^{2n} \# g(S^n \times S^n)$$



one has

Theorem. (Madsen–Weiss '07 $2n = 2$, Galatius–R-W '14 $2n \geq 4$)

$$\lim_{g \to \infty} H^*(BDiff_\partial(W_{g,1}); \mathbb{Q}) = \mathbb{Q}[\kappa_c \mid c \in \mathcal{B}]$$

Here $\mathcal{B}$ is the set of monomials in $e, p_{n-1}, p_{n-2}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$.

(For $2n \neq 4$ there is also a "stability theorem" saying how quickly the limit is attained.)

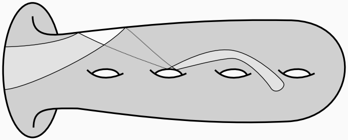
Destabilising

As $D^{2n} = W_{0,1}$, to understand $BDiff_\partial(D^{2n})$ one can try to reverse the effect of stabilising.

The crucial insight in this direction is due to Weiss, who observed that there is a fibre sequence

$$BDiff_\partial(D^{2n}) \longrightarrow BDiff_\partial(W_{g,1}) \longrightarrow BEmb^{\cong}_{\partial/2}(W_{g,1}).$$

The rightmost term consists of self-embeddings of $W_{g,1}$ which are not required to be the identity on the boundary, but only on half of the boundary.



Because of the change of boundary conditions, these embeddings have "codimension n" from the point of view of embedding theory. If $n \geq 3$ this space is therefore accessible using the Goodwillie–Weiss "calculus of embeddings".

This is how one gets started...

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