Infinite loop spaces and positive scalar curvature

Oscar Randal-Williams

University of Cambridge

September 9, 2013

joint work with Boris Botvinnik and Johannes Ebert

Positive scalar curvature

For a Riemannian manifold (M, g) of dimension d, the scalar curvature is a function $s : M \to \mathbb{R}$ whose value at a point $p \in M$ may be described by

$$\frac{volB_r(p, M)}{volB_r(0, \mathbb{R}^d)} = 1 - \frac{s(p)}{6(d+2)} r^2 + O(r^4).$$

Thus the metric g has *positive scalar curvature* (psc) if every small enough geodesic ball has smaller volume than a euclidean ball of the same radius.

If (M, g) is equipped with a Spin structure $\mathfrak{s}$, then it has a (real) spinor bundle $\not{\mathfrak{S}}_M \to M$ and an Atiyah–Singer–Dirac operator $\not{\mathfrak{D}}_g$ operating on $H = L^2(M; \not{\mathfrak{S}}_M)$. This has an index

$$\mathrm{ind}(\not{\mathfrak{D}}_g) \in KO^{-d}(*).$$

Theorem (Lichnerowicz, Hitchin)

If g is psc then the operator $\not{\mathfrak{D}}_g$ is invertible, and so $\mathrm{ind}(\not{\mathfrak{D}}_g) = 0$.

Positive scalar curvature

The index $\mathrm{ind}(\not{D}_g)$ is in fact independent of the metric g: it is the image of the spin cobordism class $[M, \mathfrak{s}]$ under the Atiyah–Bott–Shapiro orientation

$$\alpha : \Omega_d^{\mathrm{Spin}} \longrightarrow KO^{-d}(*) = KO_d(*).$$

Thus the above theorem gives a topological obstruction to admitting a psc metric.

In the early 90's, Stolz showed, using work of Gromov–Lawson, Schoen–Yau, and others, that for a simply-connected Spin manifold of dimension $d \geq 5$, this is the *only* obstruction to admitting a psc metric.

I would like to address a somewhat different question.

Suppose M does admit a psc metric: what is the (algebraic) topology of the space $\mathcal{R}^+(M)$ of all psc metrics on M?

The index difference

The first results on this question are due to Hitchin, essentially via the following construction. Let M^d be a Spin manifold.

$$\begin{array}{ccc} \mathcal{R}^+(M) & \xrightarrow{\text{ind}} & \mathrm{Fred}^{d,0}(H)^\times \simeq * \\ \downarrow & & \downarrow \\ * \simeq \mathcal{R}(M) & \xrightarrow{\text{ind}} & \mathrm{Fred}^{d,0}(H) \simeq \Omega^d(\mathbb{Z} \times BO) \end{array}$$

which gives a map

$$H(g_0) : \mathcal{R}^+(M) \longrightarrow \Omega^{d+1}(\mathbb{Z} \times BO)$$

(depending on a choice of $g_0 \in \mathcal{R}^+(M)$). By considering the composition

$$\mathrm{Diff}(M) \xrightarrow{\varphi \mapsto \varphi^* g_0} \mathcal{R}^+(M) \xrightarrow{H(g_0)} \Omega^{d+1}(\mathbb{Z} \times BO)$$

Hitchin showed that $\pi_0(H(g_0))$ can be non-trivial, and hence that $\mathcal{R}^+(M)$ can be disconnected.

Computational results, I

The coarsest of our computational results is the following.

Theorem (Botvinnik–Ebert–R-W)

For every Spin manifold $(M, \mathfrak{s})$ of dimension $d \geq 6$ with a choice of psc metric g_0, the map

$$\pi_k(\mathcal{R}^+(M)) \longrightarrow \pi_{k+d+1}(\mathbb{Z} \times BO) = \begin{cases} \mathbb{Z} & k+d+1 \equiv 0,4 \ (8) \\ \mathbb{Z}/2 & k+d+1 \equiv 1,2 \ (8) \\ 0 & \textit{else} \end{cases}$$

induced by $\pi_k(H(g_0))$ is non-zero whenever the target group is non-zero.

i.e. this map hits all the $\mathbb{Z}/2$'s, and hits a nontrivial subgroup of all the $\mathbb{Z}$'s.

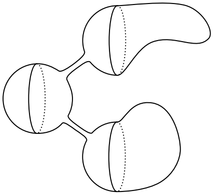
This result includes and extends those of Crowley–Schick (who prove surjectivity of this map for $d \geq 7$ and $k+d+1 \equiv 2(8)$) and those of Hanke–Schick–Steimle (who prove the mod torsion part of this theorem for $k \gg d$).

Cobordism invariance and multiplicative structures

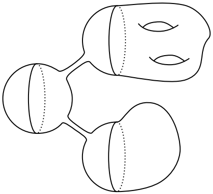
Theorem (Chernysh, Walsh)

For an embedding $D^k \times N^{d-k} \hookrightarrow M$ with $k \geq 3$, and a standard metric on $D^k \times N^{d-k}$, the inclusion $\mathcal{R}^+(M, D^k \times N^{d-k}) \hookrightarrow \mathcal{R}^+(M)$ is a weak homotopy equivalence.

In particular, $\mathcal{R}^+(M, D^d) \hookrightarrow \mathcal{R}^+(M)$ is a weak homotopy equivalence.



$\mathcal{R}^+(S^d, D^d)$ an H-space



$\mathcal{R}^+(S^d, D^d)$ acts on $\mathcal{R}^+(M^d, D^d)$

This allows us to pass from results about S^d to results about all d-manifolds.

Computational results, II

Let $g_\circ$ be the round metric on S^d, and $\mathcal{R}^+_\circ(S^d, D^d)$ be its path component. This is a (connected) H-space, and we may form its homotopical localisation $\mathcal{R}^+(S^d)_{(p)}$ at a prime p.

Theorem (Botvinnik–Ebert–R-W)

Let $d \geq 6$ and p be an odd prime. Then there is a map

$$f : \Omega_0^{d+1}(\mathbb{Z} \times BO)_{(p)} \longrightarrow \mathcal{R}^+_\circ(S^d, D^d)_{(p)}$$

such that $H(g_\circ)_{(p)} \circ f$ induces multiplication by $\mathrm{Num}(B_n/2n)$ times a p-local unit on π_{4n-d-1}.

In particular, if p is a regular prime then $H(g_\circ)_{(p)}$ is a split epimorphism and so there is a splitting of spaces

$$\mathcal{R}^+_\circ(S^d, D^d)_{(p)} \simeq \Omega_0^{d+1}(\mathbb{Z} \times BO)_{(p)} \times X.$$

There is also a result at the prime 2, but it is more complicated to state.

Geometric results

These calculations are a consequence of a more geometric result. Let

$$\theta^n : BO(2n)\langle n \rangle \to BO(2n)$$

denote the n-connected cover, and $\theta^*\gamma$ the $2n$-dimensional vector bundle classified by θ. There is a Thom spectrum $\mathbf{MT}\theta^n = \mathbf{Th}(-\theta^*\gamma)$ with associated infinite loop space $\Omega^\infty \mathbf{MT}\theta^n$.

Theorem (Botvinnik–Ebert–R-W)

There is a map

$$\psi_{g_\circ} : \Omega^{\infty+1}\mathbf{MT}\theta^n \longrightarrow \mathcal{R}^+(S^{2n})$$

such that the composition

$$H(g_\circ) \circ \psi_{g_\circ} : \Omega^{\infty+1}\mathbf{MT}\theta^n \longrightarrow \Omega^{2n+1}(\mathbb{Z} \times BO)$$

is the infinite loop map of the KO-theory Thom class of $\mathbf{MT}\theta^n$ *(up to phantom maps).*

The calculational results follow from this by pure (but involved) homotopy theory.

Words about the proof

(i) The parameterised Gromov–Lawson construction of Chernysh or Walsh shows that $\mathcal{R}^+(-)$ is cobordism invariant for simply connected Spin manifolds. Hence we may replace S^{2n} by $W_g^{2n} = \#^g S^n \times S^n$ for arbitrarily large g.

(ii) The Pontrjagin–Thom construction provides a map

$$\alpha_g : B\mathrm{Diff}(W_g^{2n}, D^{2n}) \longrightarrow \Omega_0^\infty \mathbf{MT}\theta^n$$

and in work with Søren Galatius I have proved that this is a homology equivalence in degrees $* \leq \frac{g-4}{2}$.

(iii) The technical heart of the proof is showing that the action of the mapping class group $\pi_0(\mathrm{Diff}(W_g, D))$ on $\mathcal{R}^+(W_g)$ in the homotopy category factors through an abelian group.

(iv) This means that the fibration sequence

$$\mathcal{R}^+(W_g) \longrightarrow \mathcal{R}^+(W_g) /\!\!/ \mathrm{Diff}(W_g, D) \longrightarrow B\mathrm{Diff}(W_g, D)$$

is pulled back from a fibration over the $+$-construction $B\mathrm{Diff}(W_g, D)^+$, which *homotopically* approximates $\Omega_0^\infty \mathbf{MT}\theta^n$ by my result with Galatius.

(v) Take the limit as $g \to \infty$ carefully.

Words about the index theory

There are some subtle points of index theory which need to be treated.

- Considering a 1-parameter family of psc metrics on S^d as a psc metric on $[0,1] \times S^d$, then closing off the ends, gives a map

$$\Omega\mathcal{R}^+(S^d) \longrightarrow \mathcal{R}^+(S^{d+1}).$$

(This is how we upgrade results about even spheres to all spheres.)

- As described earlier, there is an "action"

$$\mathcal{R}^+(S^d, D^d) \times \mathcal{R}^+(M^d, D^d) \longrightarrow \mathcal{R}^+(M^d, D^d).$$

In both cases we would like the evident diagram involving the secondary index maps H to commute. There is an alternative definition of the secondary index with which it is easier to prove these results, but these definitions have been shown to be equivalent in

J. Ebert, *The two definitions of the index difference*, 2013, arXiv:1308.4998.