Homological stability for moduli spaces of manifolds

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(joint work with Søren Galatius)

Fix a dimension $2n$ and consider the smooth closed $2n$-dimensional manifold $W_g := \#^g S^n \times S^n$. Choosing once and for all an embedding $D^{2n} \hookrightarrow W_g$, we can form the manifold with boundary

$$W_{g,1} := W_g \setminus \mathrm{int}(D^{2n}).$$

Let $\mathrm{Diff}_\partial(W_{g,1})$ denote the topological group of diffeomorphisms of $W_{g,1}$ which are the identity on a neighbourhood of the boundary. A choice of embedding $W_{g,1} \hookrightarrow W_{g+1,1}$ gives a continuous homomorphism $\mathrm{Diff}_\partial(W_{g,1}) \to \mathrm{Diff}_\partial(W_{g+1,1})$, and so a map $\mathcal{S}$ on classifying spaces.

In my talk I presented the proof of the following theorem, from [2].

Theorem A. *Suppose that $2n > 4$. Then the induced map*

$$\mathcal{S}_* : H_*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z}) \longrightarrow H_*(B\mathrm{Diff}_\partial(W_{g+1,1}); \mathbb{Z})$$

on integral homology is an isomorphism in degrees $* \leq \frac{g-4}{2}$.

This theorem is also true for $2n < 4$ (though with different stability ranges). If $2n = 0$, it is Nakaoka's stability theorem [5] for the homology of symmetric groups. If $2n = 2$, it is Harer's stability theorem [4] for the homology of mapping class groups of oriented surfaces.

Remark 1. Independently, Berglund and Madsen [1] have obtained a result similar to Theorem A, for rational cohomology in the range $* \leq \min(n-3, (g-6)/2)$. (for details see the contribution of A. Berglund to this volume.)

Our motivation for proving Theorem A is that in previous work [3] we have identified the ring

$$\varprojlim_{g \to \infty} H^*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Z})$$

with the cohomology of an explicit infinite loop space, which allows for concrete calculations to be made. (This is too involved to explain here, but see [6] for a precis.) Along with Theorem A, this allows us to obtain interesting cohomological information about $H^*(B\mathrm{Diff}_\partial(W_{g,1}))$ in degrees $* \leq \frac{g-4}{2}$.

References

[1] Alexander Berglund and Ib Madsen, *Homological stability of diffeomorphism groups*, arXiv:1203.4161, 2012.

[2] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds*, arXiv:1203.6830, 2012.

[3] ———, *Stable moduli spaces of high dimensional manifolds*, arXiv:1201.3527, 2012.

[4] John L. Harer, *Stability of the homology of the mapping class groups of orientable surfaces.* Ann. of Math. (2), 121(2):215–249, 1985.

[5] Minoru Nakaoka, *Decomposition theorem for homology groups of symmetric groups.* Ann. of Math. (2), 71:16–42, 1960.

[6] Oscar Randal-Williams, *Monoids of moduli spaces of manifolds, II*, Oberwolfach Reports. Vol. 7, no. 3, 2484–2486, 2010.