Stable moduli spaces of high dimensional manifolds

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(joint work with Søren Galatius)

For a closed smooth manifold W, we consider the *moduli space of manifolds of type* W to be the classifying space $B\mathrm{Diff}(W)$ of the group of diffeomorphisms of W. One justification for this name is that $B\mathrm{Diff}(W)$ carries a smooth fibre bundle with fibre W, and this is the universal example of such a bundle, i.e. any such bundle $\pi : E \to B$ is obtained up to isomorphism by pulling back this universal bundle along a unique homotopy class of map $f : B \to B\mathrm{Diff}(W)$. Hence the cohomology ring $H^*(B\mathrm{Diff}(W))$ is precisely the ring of characteristic classes of smooth fibre bundles with fibre W.

Suppose that W has dimension $2n$, and define the *genus* of W by

$$g(W) := \max\{g \in \mathbb{N} \mid \#^g S^n \times S^n \text{ is a connect-summand of } W\}.$$

When $2n = 2$ and W is orientable, this coincides with the usual genus of a surface. The tangent bundle of W is classified by a Gauss map $\tau_W : W \to BO(2n)$, and we may form the Moore–Postnikov n-stage of this map,

$$\tau_W : W \xrightarrow{\ell} B \xrightarrow{\theta} BO(2n),$$

where θ is a fibration. Recall that this is a factorisation of τ_W having the property that ℓ is n-connected and θ is n-co-connected. It is characterised up to homotopy equivalence over $BO(2n)$ by these properties.

We may form the following two objects associated to the fibration θ. Firstly, let $\mathbf{MT}\theta$ be the Thom spectrum associated to the virtual vector bundle $-\theta^*\gamma_{2n}$ over B. Secondly, let $\mathrm{hAut}(\theta)$ denote the grouplike topological monoid of self-homotopy equivalences of B over $BO(2n)$, i.e. homotopy equivalences $f : B \overset{\sim}{\to} B$ such that $\theta \circ f = \theta$. As the Thom spectrum construction is functorial, we obtain an action of $\mathrm{hAut}(\theta)$ on the spectrum $\mathbf{MT}\theta$, and hence on its associated infinite loop space $\Omega^\infty\mathbf{MT}\theta$. The set of path components $\pi_0(\Omega^\infty\mathbf{MT}\theta)$ has a cobordism-theoretic description—via the Pontrjagin–Thom construction—in terms of $2n$-dimensional manifolds equipped with a lift of their Gauss map along the fibration θ, and we let $\Omega^\infty_{[W]}\mathbf{MT}\theta$ be the union of those path components given by the $\mathrm{hAut}(\theta)$-orbit of $[W, \ell] \in \pi_0(\Omega^\infty\mathbf{MT}\theta)$.

With this preparation, our main theorem is as follows, which extends the Madsen–Weiss theorem and related homological stability results when $2n = 2$.

Theorem A. *There is a map*

$$\alpha_W : B\mathrm{Diff}(W) \longrightarrow \big(\Omega^\infty_{[W]}\mathbf{MT}\theta\big)/\!\!/\mathrm{hAut}(\theta),$$

which, if W is simply-connected and $2n \geq 6$, induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$.

Many variations of this theorem also hold: there is a version for orientation-preserving diffeomorphisms, where one replaces θ by a lift $\theta^+ : B \to BSO(2n)$ to the classifying space for oriented $2n$-dimensional vector bundles; there is a version for manifolds W with non-empty boundary, where the target of α_W is again modified slightly.

As the target of the map α_W is constructed in purely homotopy-theoretic terms, it is amenable to calculation using the traditional tools of algebraic topology. In particular, for various interesting manifolds (such as $\#^g S^n \times S^n$, or smooth hypersurfaces in $\mathbb{CP}^4$) one can now compute the ring $H^*(B\mathrm{Diff}(W);\mathbb{Q})$ in this stable range of degrees.

Theorem A is a consequence of a collection of more technical results, spread throughout [1, 2, 3]. However, these more technical results are of independent interest, and for theoretical rather than computational applications may be more useful than Theorem A itself. (See J. Ebert's report in this volume for an example.)

Moduli spaces of θ-manifolds. Rather than constructing the fibration θ from the manifold W, we may take a different point of view: fix an n-co-connected fibration $\theta : B \to BO(2n)$, and consider the space of all n-connected maps $W \to B$ which are lifts of τ_W along the fibration θ (and are specified on ∂W). Alternatively, we may fix a bundle map $\ell_\partial : TW|_{\partial W} \to \theta^*\gamma_{2n}$ and consider the homotopy equivalent space

$$\mathrm{Bun}_n^\theta(W) := \{\ell : TW \to \theta^*\gamma_{2n} \text{ an } n\text{-connected bundle map extending } \ell_\partial\}.$$

From this we may form the Borel construction

$$B\mathrm{Diff}_\partial^\theta(W) := \mathrm{Bun}_n^\theta(W)/\!\!/\mathrm{Diff}_\partial(W),$$

a moduli space of θ-manifolds of type W. When $\partial W = \emptyset$ the monoid $\mathrm{hAut}(\theta)$ can be made to act on this space, and there is a homotopy equivalence

$$B\mathrm{Diff}^\theta(W)/\!\!/\mathrm{hAut}(\theta) \simeq B\mathrm{Diff}(W).$$

Hence Theorem A is a consequence of the following theorem, which we now formulate for manifolds with boundary.

Theorem B. *There is a map*

$$\alpha_W^\theta : B\mathrm{Diff}_\partial^\theta(W) \longrightarrow \Omega^\infty_{[W]}\mathbf{MT}\theta,$$

which, if W is simply-connected and $2n \geq 6$, induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$.

Theorem A is obtained by taking Borel constructions of both sides by $\mathrm{hAut}(\theta)$, using the fact that α_W^θ may be chosen to be $\mathrm{hAut}(\theta)$-equivariant.

Homology stability with respect to $S^n \times S^n$. An immediate consequence of Theorem B is the fact that $B\mathrm{Diff}^\theta(W)$ and $B\mathrm{Diff}^\theta(W \# S^n \times S^n)$ have the same homology in the stable range of degrees, as they both have the homology of a collection of path components of $\Omega^\infty\mathbf{MT}\theta$. When W has non-empty boundary there is a *stabilisation map*

$$B\mathrm{Diff}_\partial^\theta(W) \longrightarrow \mathrm{Diff}_\partial^\theta(W \# S^n \times S^n)$$

inducing this homology isomorphism, given by gluing on $([0,1] \times \partial W)\#(S^n \times S^n)$ with some θ-structure. This is an independent ingredient of these theorems, and holds in greater generality. The following theorem is proved in [2].

Theorem C. *Let $\theta : B \to BO(2n)$ be spherical (every θ-structure on D^{2n} extends to S^{2n}), but not necessarily n-co-connected. Let $B\mathrm{Diff}^{\theta}_{\partial}(W)$ be defined as above, but using all bundle maps, not just the n-connected ones. Then the stabilisation map induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$ as long as W is simply-connected and $2n \geq 6$.*

Homology stability with respect to higher handles. A further immediate consequence of Theorem B is an analogous homological stability theorem for gluing on to W a θ-cobordism $K : \partial W \rightsquigarrow P$ such that $(K, \partial W)$ is $(n-1)$-connected, i.e. attaching to W handles of index n or higher. This is again an independent ingredient; given Theorem C can be phrased as follows, which will appear in [3].

Theorem D. *Let $\theta : B \to BO(2n)$ be n-co-connected, partition $\partial W = Q \cup D^{2n-1}$, and let $K : Q \rightsquigarrow Q'$ be a θ-cobordism which is trivial on the boundary and such that (K, Q) is $(n-1)$-connected. Let $S : D^{2n-1} \rightsquigarrow D^{2n-1}$ be $([0,1] \times D^{2n-1})\#(S^n \times S^n)$. Then the map*

$$- \cup K : \mathrm{hocolim}_k B\mathrm{Diff}^{\theta}_{\partial}(kS \cup W) \longrightarrow \mathrm{hocolim}_k B\mathrm{Diff}^{\theta}_{\partial}(kS \cup W \cup K)$$

induces an isomorphism on homology as long as $2n \geq 4$.

Stable homology. A further immediate consequence of Theorem B is that the induced map after stabilising by $S := ([0,1] \times \partial W)\#(S^n \times S^n)$,

$$\mathrm{hocolim}_k B\mathrm{Diff}^{\theta}_{\partial}(kS \cup W) \longrightarrow \mathrm{hocolim}_k \Omega^{\infty}_{[kS \cup W]} \mathbf{MT}\theta,$$

is a homology equivalence (as $g(kS \cup W) \geq k + g(W)$ diverges). This again holds in much greater generality: θ can be just spherical rather than n-co-connected; W need not be simply connected; we can take $B\mathrm{Diff}^{\theta}$ in the sense given in Theorem C (though in this generality we may need to stabilise by more than just S). A statement of the general result is complicated, and we refer to [1] for a detailed statement, and the proof.

References

[1] S. Galatius, O. Randal-Williams, *Stable moduli spaces of high dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377.

[2] S. Galatius, O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. I*, arXiv:1403.2334.

[3] S. Galatius, O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. II*, in preparation, $\geq$ 2014.