On Rognes' connectivity conjecture

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(joint work with Søren Galatius, Alexander Kupers)

In 1992 Rognes [2] introduced a filtration $F_\bullet\mathbb{K}(R)$ of the (free) algebraic K-theory spectrum $\mathbb{K}(R)$ of a ring R having the invariant basis number property. He identified the filtration quotients as the homotopy orbits

$$\frac{F_n\mathbb{K}(R)}{F_{n-1}\mathbb{K}(R)} \simeq \mathbb{D}(R^n)/\!\!/GL(R^n)$$

for a certain $GL(R^n)$-spectrum $\mathbb{D}(R^n)$. The kth space in this spectrum, $D^k(R^n)$, is called the k-dimensional building. When $k = 1$ it is the double suspension of the Tits building of R^n, and if R is a field then it follows from the Solomon–Tits theorem that $D^1(R^n)$ is a wedge of n-spheres, so in particular is $(n-1)$-connected. Based on detailed calculations, Rognes conjectured that $\mathbb{D}(R^n)$ is $(2n-3)$-connected when R is either local or Euclidean, and he verified this for $n = 2$.

If this property held, then the homotopy orbits $\mathbb{D}(R^n)/\!\!/GL(R^n)$ would also be $(2n-3)$-connected. This would mean that the spectral sequence associated to this filtration would converge rather more quickly than might be expected. While we are not able to settle Rognes' conjecture as he stated it, we are able to show that it holds for infinite fields after taking homotopy orbits: the fast convergence follows.

Theorem A. *If R is an infinite field then $\mathbb{D}(R^n)/\!\!/GL(R^n)$ is $(2n-3)$-connected.*

The majority of this talk was an outline of the proof, which is quite elementary; from now onwards suppose that R is a field. The k-dimensional buildings $D^k(R^n)$ are first compared with certain "split" k-dimensional buildings $\tilde{D}^k(R^n)$, which relate to direct-sum K-theory as the $D^k(R^n)$ relate to exact-sequence K-theory. There is a canonical $GL(R^n)$-equivariant map

$$\tilde{D}^k(R^n) \longrightarrow D^k(R^n),$$

arising from the inclusion of split exact sequences into all exact sequences. This is not an equivalence, but it induces a bijection on $GL(R^n)$-orbits of simplices in each degree, and the maps on stabiliser groups are k-fold analogues of inclusions

$$\{\text{block diagonal matrices}\} \longrightarrow \{\text{block upper-triangular matrices}\}.$$

This is of course not an isomorphism of groups, but it follows from a remarkable theorem of Nesterenko–Suslin [1] that—as long as R has "many units", which an *infinite* field does—the map induces an isomorphism on integral homology. It follows that the maps on pointed homotopy orbits

$$\tilde{D}^k(R^n)/\!\!/GL(R^n) \longrightarrow D^k(R^n)/\!\!/GL(R^n)$$

are stable homotopy equivalences, so after taking homotopy-orbits the split and non-split k-dimensional buildings may be freely interchanged.

The advantage of the $\tilde{D}^k(R^n)$ over the $D^k(R^n)$ is that the $(k+1)$-st can be produced from the k-th by a bar construction. In particular, it is easy to show that if the $\tilde{D}^k(R^n)/\!\!/GL(R^n)$ are $(2n-3+k)$-connected for all n, then the $\tilde{D}^{k+1}(R^n)/\!\!/GL(R^n)$ are $(2n-3+k+1)$-connected for all n. To see the required connectivity of $\mathbb{D}(R^n)/\!\!/GL(R^n)$ it therefore suffices to show that $D^2(R^n)$ is $(2n-1)$-connected, but this may be shown to be homeomorphic to $D^1(R^n) \wedge D^1(R^n)$, so a wedge of $2n$-spheres by the Solomon–Tits theorem.

References

[1] Yu. P. Nesterenko, A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.

[2] J. Rognes, *A spectrum level rank filtration in algebraic K-theory*, Topology **31** (1992), no. 4, 813-845.