“GROUP-COMPLETION”, LOCAL COEFFICIENT SYSTEMS, AND PERFECTION

by

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Abstract. — It has been known since the foundation of higher algebraic K-theory that certain maps which are shown to induce isomorphisms on homology by “group-completion” arguments are in fact acyclic maps, i.e. induce isomorphisms on homology with all systems of local coefficients. This strengthening, in the cases where it is proved, is usually established by an *ad hoc* argument.

In this note we give a systematic treatment of this acyclicity question by supplying the proof of an assertion made by McDuff and Segal in their proof of the “group-completion” theorem. Specifically, if M is a homotopy commutative topological monoid and M_∞ is a cofinal (right) stabilisation, we explain why the comparison map $s : M_\infty \to \Omega BM$ is always an acyclic map.

1. Introduction

This note is a scholium on *Homology Fibrations and the “Group-Completion” Theorem*, by Dusa McDuff and Graeme Segal, hereafter called [**5**]. As the present note is not intended to be read in isolation, we freely use the language and notation of [**5**] without further explanation. Our aim is to explicate certain remarks made in that paper related to systems of local coefficients, and in particular to give a proof of the following theorem.

Theorem 1.1. — *Let M be a homotopy commutative topological monoid, and denote by $[x]$ the path component of an element $x \in M$. Let $m_1, m_2, m_3, \ldots \in M$ be a sequence of elements such that for every $m \in M$ and $n \in \mathbb{N}$, there exists a $k \geq 0$ such that $[m]$ is a right factor of $[m_{n+1} \cdot m_{n+2} \cdots m_{n+k}]$ in the discrete monoid $\pi_0(M)$. Form*

$$M_\infty := \mathrm{hocolim}(M \overset{\cdot m_1}{\to} M \overset{\cdot m_2}{\to} M \overset{\cdot m_3}{\to} \cdots).$$

Then the McDuff–Segal comparison map

$$M_\infty \overset{s}{\longrightarrow} \mathrm{hofib}_*(\pi) \underset{\sim}{\overset{t}{\longleftarrow}} \Omega BM$$

induces an isomorphism on homology with all systems of local coefficients on ΩBM.

Before starting it is worth mentioning precisely what we mean by the McDuff–Segal comparison map. There is no especially natural comparison map directly between M_∞ and ΩBM, so instead we use the following construction. The monoid M acts on the left of M_∞, so we can form the Borel construction

$$\pi : EM \times_M M_\infty \longrightarrow BM.$$

Commuting homotopy colimits, we can write $EM \times_M M_\infty$ as a mapping telescope of self-maps of the weakly contractible space $EM \times_M M$, so it is weakly contractible. Thus the homotopy fibre of π over the basepoint is weakly homotopy equivalent to ΩBM, and the actual fibre is M_∞. We obtain comparison maps

$$M_\infty \xrightarrow{s} \mathrm{hofib}_*(\pi) \underset{\sim}{\xleftarrow{t}} \Omega BM,$$

where s compares the fibre of π with its homotopy fibre, and t compares the homotopy fibre of $* \to BM$ with that of π. These are the maps to which the theorem refers.

1.1. Perfection. — A consequence of Theorem 1.1 is that the homotopy fibre F of $s : M_\infty \to \mathrm{hofib}_*(\pi)$ is acyclic, and so its fundamental group is perfect. Taking the long exact sequence on homotopy groups based at the basepoint $e \in M \subset M_\infty$ gives

$$\cdots \longrightarrow \pi_1(F, e) \longrightarrow \pi_1(M_\infty, e) \longrightarrow \pi_1(\mathrm{hofib}_*(\pi), s(e)) \longrightarrow *,$$

and as $\pi_1(\mathrm{hofib}_*(\pi), s(e))$ is the abelianisation of $\pi_1(M_\infty, e)$ (it is certainly abelian, as $\mathrm{hofib}_*(\pi) \simeq \Omega BM$, and s is a homology equivalence) we see that the commutator subgroup of $\pi_1(M_\infty, e)$ is a quotient of the perfect group $\pi_1(F, e)$, so also perfect.

The above argument is explained in [**5**, Remark 2], where it is claimed that "If isolated this would reduce to Wagoner's argument in [**7**]." Although [**7**] certainly contains related results, we were unable to find a general proof of the perfection of the commutator subgroup of M_∞ in such a situation. We give a separate proof of this interesting property in §3.

Corollary 1.2. — *Let M, $m_1, m_2, \ldots$ satisfy the assumptions of Theorem 1.1. Then there is a weak homotopy equivalence*

$$M_\infty^+ \simeq \Omega BM,$$

where M_∞^+ is the space obtained from M_∞ by forming the Quillen plus-construction of each path component with respect to the commutator subgroup of its fundamental group.

Proof. — The map $s : M_\infty \to \mathrm{hofib}_*(\pi)$ has target equivalent to a loop space, so extends over the plus-construction to a map $s^+ : M_\infty^+ \to \mathrm{hofib}_*(\pi)$. This map induces an isomorphism on homology with all local coefficients (by Theorem 1.1, as $M_\infty \to M_\infty^+$ is an acyclic map) and on fundamental groups with any basepoint (as both spaces have abelian fundamental group), so is a weak homotopy equivalence. □

Remark 1.3. — For the monoids $M = \coprod_{[P]} B\mathrm{Aut}(P)$ occurring in the algebraic K-theory of rings, the result of Corollary 1.2 is usually proved directly by establishing the structure of an H-space on M_∞^+; then $M_\infty^+ \to \Omega BM$ is a homology equivalence between simple spaces, hence a homotopy equivalence. The details of this line of argument are explained admirably in [**7**], and seems to be the line of argument McDuff and Segal had in mind, but the argument of [**7**] relies of methods of group homology and is not directly applicable to general topological monoids. We prefer to prove Theorem 1.1 directly, by verifying an implicit assumption in [**5**].

1.2. Example: the theorem of Barratt–Priddy, Quillen, and Segal. — Consider the monoid $M := \coprod_{n \geq 0} B\Sigma_n$, with multiplication coming from the homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. It is well known [**1**] that there is a homotopy equivalence between the group completion ΩBM and the space $QS^0 := \mathrm{colim}_{k \to \infty} \Omega^k S^k$, and so by Theorem 1.1 an acyclic map

$$s : \mathbb{Z} \times B\Sigma_\infty \longrightarrow QS^0,$$

and by Corollary 1.2 a homotopy equivalence $B\Sigma_\infty^+ \simeq Q_0 S^0$, to the basepoint component of QS^0.

There is a coefficient system A over Q_0S^0 whose fibre at the basepoint $*$ is the $\pi_1(Q_0S^0, *)$-module $\mathbb{Z}[\pi_1(Q_0S^0, *)]$. There is an isomorphism $\pi_1(Q_0S^0, *) \cong \mathbb{Z}/2$, and the map

$$\Sigma_\infty = \pi_1(B\Sigma_\infty, *) \longrightarrow \pi_1(Q_0S^0, *) = \mathbb{Z}/2$$

is the sign homomorphism, so s^*A is the coefficient system over $B\Sigma_\infty$ corresponding to the sign representation. Theorem 1.1 applies and gives the sequence of isomorphisms

$$H_*(BA_\infty; \mathbb{Z}) \cong H_*(B\Sigma_\infty; s^*A) \xrightarrow{\simeq} H_*(Q_0S^0; A),$$

which identifies the homology of the stable alternating group with the homology of the universal cover of Q_0S^0.

As there is a splitting $Q_0S^0 \simeq \widetilde{Q_0S^0} \times K(\mathbb{Z}/2, 1)$ of spaces (see e.g. [**3**, Lemma 7.1]), we find that $H^*(A_\infty; \mathbb{F}_p) \cong H^*(\Sigma_\infty; \mathbb{F}_p)$ for p odd, and that $H^*(A_\infty; \mathbb{F}_2) \cong H^*(\Sigma_\infty; \mathbb{F}_2) \otimes_{\mathbb{F}_2[x]} \mathbb{F}_2$, for $\mathbb{F}_2[x] \cong H^*(\mathbb{Z}/2; \mathbb{F}_2) \to H^*(\Sigma_\infty; \mathbb{F}_2)$ the map on cohomology induced by the sign homomorphism.

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2. Proof of Theorem 1.1

Suppose that $A \to \mathrm{hofib}_*(\pi)$ is a system of local coefficients such that

Assumption 2.1. — For each $m \in M$ the map

$$(m \cdot -)_* : H_*(M_\infty; (m \cdot -)^* s^*A) \longrightarrow H_*(M_\infty; s^*A)$$

is an isomorphism.

Then McDuff–Segal's argument shows that

$$s_* : H_*(M_\infty; s^*A) \longrightarrow H_*(\mathrm{hofib}_*(\pi); A)$$

is an isomorphism, and so $H_*(M_\infty; s^*A) \cong H_*(\Omega BM; t^*A)$. In fact, the crucial proposition [**5**, Proposition 6] is not quite correct, but corrections have been written by McDuff in [**4**, Lemma 3.1]. We will verify Assumption 2.1 in the situation described in Theorem 1.1

Remark 2.2. — There exist coefficient systems $B \to M_\infty$ such that

$$(m \cdot -)_* : H_*(M_\infty; (m \cdot -)^*B) \longrightarrow H_*(M_\infty; B)$$

is not an isomorphism, but they are not pulled back from $\mathrm{hofib}_*(\pi)$. We give an example of such a coefficient system in Remark 2.6. The key property that coefficient systems of the form s^*A have is that they are *abelian*: the monodromy around any commutator is zero.

2.1. Abelian coefficient systems. — Let X be a space and $p : A \to X$ a local coefficient system, i.e. a bundle of abelian groups.

Definition 2.3. — We say $p : A \to X$ is *abelian* if for every $x \in X$ the action of the group $\pi_1(X, x)$ on the fibre $A_x := p^{-1}(x)$ factors through the abelianisation of $\pi_1(X, x)$.

Suppose that X is path-connected and $p : A \to X$ is an abelian local coefficient system, and let $\alpha \in H_1(X; \mathbb{Z})$ be given. We construct a bundle map

$$T_\alpha : A \to A$$

covering the identity map of X, as follows. For a point $a \in A_x$ choose a loop $\gamma \in \pi_1(X, x)$ representing the homology class α, then lift the path γ starting at a. Define $T_\alpha(a) \in A_x$ to be the end point of this path.

Lemma 2.4. — *The map T_α is well-defined.*

Proof. — The only ambiguity is the choice of path: another choice γ' differs from γ by an element c in the kernel of the abelianisation, i.e. in the derived subgroup of $\pi_1(X, x)$. As this loop c must lift to a loop in A, the end points of the lifts of the paths γ and γ' are equal. □

Remark 2.5. — If $x \in X$ is a chosen basepoint, and X is path-connected, there is a correspondence between local coefficient systems over X and $\mathbb{Z}[\pi_1(X, x)]$-modules. Similarly, without a choice of basepoint there is a correspondence between abelian local coefficients systems over X and $\mathbb{Z}[H_1(X; \mathbb{Z})]$-modules. From this point of view, the map T_α corresponds to the automorphism of a $\mathbb{Z}[H_1(X; \mathbb{Z})]$-module given by multiplication by α, which is a $\mathbb{Z}[H_1(X; \mathbb{Z})]$-module map as $H_1(X; \mathbb{Z})$ is abelian.

2.2. Set-up. — Let $p : A \to \mathrm{hofib}_*(\pi)$ be a system of local coefficients. As there is an equivalence $\mathrm{hofib}_*(\pi) \simeq \Omega BM$, the fundamental group of $\mathrm{hofib}_*(\pi)$ at any basepoint is abelian, and so the system of coefficients A is abelian in the sense of Definition 2.3.

Let us define $B := s^*A \to M_\infty$, another abelian system of coefficients, and for $m \in M$ consider the map

$$(m \cdot -)_* : H_*(M_\infty; (m \cdot -)^* B) \longrightarrow H_*(M_\infty; B) \tag{2.1}$$

which we want to show is an isomorphism. Let us write $M_n \subset M_\infty$ for the nth copy of M in the mapping telescope, and $B_n \to M_n$ for the pullback of B to M_n.

2.3. (2.1) is surjective. — Let $\alpha \in H_*(M_\infty; B)$ be a class, which we may suppose is represented by $\alpha' \in H_*(M_n; B_n)$. By assumption, there is a factorisation $[m_{n+1} \cdot m_{n+2} \cdots m_{n+k}] = [x] \cdot [m]$ for some $k \gg 0$. Using this and homotopy commutativity, we see that the maps

$$(- \cdot m_{n+1} \cdot m_{n+2} \cdots m_{n+k}), (m \cdot - \cdot x) : M \to M$$

are homotopic. Let $G : I \times M \to M$ be a homotopy from $- \cdot m_{n+1} \cdot m_{n+2} \cdots m_{n+k}$ to $m \cdot - \cdot x$. We then have a commutative diagram as follows.

$$\begin{array}{ccc}
H_*(\{0\} \times M; B_n) & \xrightarrow{(-\cdot m_{n+1}\cdot m_{n+2}\cdots m_{n+k})_*} & H_*(M; B_{n+k}) \\
\downarrow \simeq & & \\
H_*(I \times M; G^* B_{n+k}) & \xrightarrow{\quad G_* \quad} & H_*(M; B_{n+k}) \\
\uparrow \simeq & \nearrow (m\cdot - \cdot x)_* & \uparrow (m\cdot -)_* \\
H_*(\{1\} \times M; (m \cdot - \cdot x)^* B_{n+k}) & \xrightarrow[(-\cdot x)_*]{} & H_*(M; (m \cdot -)^* B_{n+k})
\end{array}$$

Then $\alpha'' = (- \cdot m_{n+1} \cdot m_{n+2} \cdots m_{n+k})_*(\alpha')$ also represents α. By the commutativity of the top triangle, this class is in the image of the map G_*, and so in the image of the composition $(m \cdot -)_* \circ (- \cdot x)_*$. In particular, it is in the image of the map $(m \cdot -)_*$, as required. □

Remark 2.6. — We did not use that the system of coefficients is abelian. This argument shows that (2.1) is surjective with *any* system of coefficients $B \to M_\infty$.

It is not the case that (2.1) will also be injective with any system of coefficients. For example, in the situation of §1.2 we have $M_\infty = \mathbb{Z} \times B\Sigma_\infty$, and we can let B be the local coefficient system which over $\{0\} \times B\Sigma_\infty$ corresponds to the module $\mathbb{Z}[X]$, where X is a free transitive Σ_∞-set, and over the other path components is the zero local coefficient system. Multiplication by the point $m = * \in B\Sigma_1 \subset \coprod_{n \geq 0} B\Sigma_n$ gives a pointed map $m \cdot - : B\Sigma_\infty \to B\Sigma_\infty$ which on fundamental groups induces a homomorphism $S : \Sigma_\infty \to \Sigma_\infty$. There is an isomorphism of Σ_∞-sets $S^*X \cong \coprod^\infty X$, and so $(m \cdot -)^*B$ is the local coefficient system which over $\{-1\} \times B\Sigma_\infty$ corresponds to $\bigoplus^\infty \mathbb{Z}[X]$ and over the other path components is the zero local coefficient system. On zeroth homology the map (2.1) is a homomorphism of the form $\bigoplus^\infty \mathbb{Z} \to \mathbb{Z}$, which cannot be injective.

2.4. (2.1) is injective. — Let $\beta \in H_*(M_\infty; (m \cdot -)^* B)$ be a class such that $(m \cdot -)_*(\beta) = 0 \in H_*(M_\infty; B)$. Then we can represent β by a $\beta' \in H_*(M_n; (m \cdot -)^* B_n)$ such that

(i) $(m \cdot -)_*(\beta') = 0 \in H_*(M_n; B_n)$,
(ii) if we write $\mathrm{supp}(\beta') \subset \pi_0(M_n)$ for the finite set of path components to which β' has a nontrivial projection, then $\mathrm{supp}(\beta')$ injects into $\pi_0(M_\infty)$.

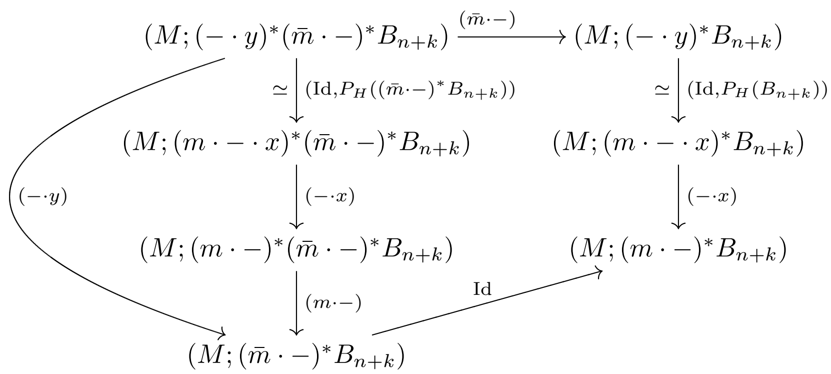
As above, there is a $k \gg 0$ such that $[m_{n+1} \cdot m_{n+2} \cdots m_{n+k}] = [x] \cdot [m]$, and we choose a homotopy $H : I \times M \to M$ from $- \cdot m_{n+1} \cdot m_{n+2} \cdots m_{n+k}$ to $m \cdot - \cdot x$. For ease of notation, let $y = m_{n+1} \cdot m_{n+2} \cdots m_{n+k}$. For any coefficient system $C \to M$, we obtain a system $H^* C \to I \times M$ and so a preferred isomorphism of coefficient systems

$$P_H(C) : (- \cdot y)^* C \longrightarrow (m \cdot - \cdot x)^* C$$

covering the identity map of M. Concretely, over a point $u \in M$ this isomorphism is given by the map $C|_{u \cdot y} \to C|_{m \cdot u \cdot x}$ obtained by lifting the path $H(-, u)$ from $u \cdot y$ to $m \cdot u \cdot x$ up the bundle $C \to M$. As in the diagram above we obtain the equation

$$(- \cdot y)_* = (m \cdot -)_* \circ (- \cdot x)_* \circ (\mathrm{Id}, P_H(C))_* : H_*(M; (- \cdot y)^* C) \longrightarrow H_*(M; C).$$

Let us apply this to the coefficient system $(\bar{m} \cdot -)^* B_{n+k}$, where $\bar{m} = m$ but the notation will help us distinguish the two copies. This shows that the left-hand portion of the following diagram commutes at the level of homology.



The strategy will now be as follows. The trapezium commutes at the level of underlying maps of spaces, but not at the level of local coefficient systems. We will construct an automorphism T of the coefficient system $(\bar{m} \cdot -)^* B_{n+k}$ so that the trapezium does commute up to this automorphism, at least when restricted to those path components of M supporting β'. As β' goes to zero clockwise around the diagram, it must also do so anticlockwise, which proves injectivity.

Let us explain how to construct the automorphism T of $(\bar{m} \cdot -)^* B_{n+k}$. The lower route around the trapezium is

$$(u \in M, a \in B_{n+k}|_{\bar{m}uy}) \longmapsto (mux, P_H((\bar{m} \cdot -)^* B_{n+k})(a) \in B_{n+k}|_{\bar{m}mux})$$

whereas the upper route is

$$(u \in M, a \in B_{n+k}|_{\bar{m}uy}) \longmapsto (mux, P_H(B_{n+k})(a) \in B_{n+k}|_{\bar{m}mux}).$$

That is, the underlying map of spaces are the same; on coefficients the lower route comes from lifting the path $\bar{m} \cdot (H(-, u) : uy \rightsquigarrow mux)$, and the upper route comes from lifting the path $H(-, \bar{m}u) : (\bar{m}u)y \rightsquigarrow m(\bar{m}u)x$. These two paths have the same start and end points, but are not necessarily homotopic: they differ by an element $\gamma_u \in \pi_1(M, \bar{m}mux)$

Recall that for a point $u \in M$ we write $[u] \subset M$ for the path component of u. Then by the discussion above, for a fixed point $u \in M$ the two maps

$$([u]; (- \cdot y)^*(\bar{m} \cdot -)^*(B_{n+k}|_{[\bar{m}uy]})) \longrightarrow ([mux]; (\bar{m} \cdot -)^*(B_{n+k}|_{[\bar{m}mux]}))$$

differ by the automorphism $(\bar{m} \cdot -)^*(T_{[\gamma_u]})$ of $(\bar{m} \cdot -)^*(B_{n+k}|_{[\bar{m}mux]})$.

We have the element β' in the homology of the top left corner. Choose for each component $\alpha \in \mathrm{supp}(\beta')$ a point u_α in that component. Let

$$T : (\bar{m} \cdot -)^* B_{n+k} \longrightarrow (\bar{m} \cdot -)^* B_{n+k}$$

be the automorphism which over the path component of $m \cdot u_\alpha \cdot x$ is $(\bar{m} \cdot -)^*(T_{[\gamma_{u_\alpha}]})$, and is the identity over components not of the form $[m \cdot u_\alpha \cdot x]$. Because the support of β' injects into $\pi_0(M_\infty)$, if $\alpha \neq \alpha'$ then $[m \cdot u_\alpha \cdot x] \neq [m \cdot u_{\alpha'} \cdot x]$, so T is well defined. This finishes the construction of T.

Finally, by construction we have

$$(- \cdot y)_*(\beta') = T \circ (- \cdot x)_* \circ (Id, P_H(B_{n+k}))_* \circ (\bar{m} \cdot -)_*(\beta').$$

We have assumed that $(\bar{m} \cdot -)(\beta') = 0$, and so $(- \cdot y)_*(\beta') = 0$ and hence $\beta = 0$. □

3. Perfection

Let M be a homotopy commutative topological monoid, and suppose that we are given an element $m_0 \in M$. Define a sequence of groups $G_n := \pi_1(M, m_0^n)$. The monoid structure defines homomorphisms

$$\mu_{n,m} : G_n \times G_m \longrightarrow G_{n+m}$$

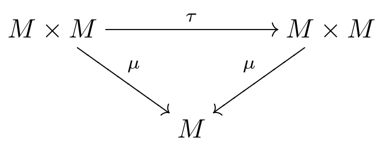
which satisfy the associativity condition

$$\mu_{n+m,k}(\mu_{n,m}(x, y), z) = \mu_{n,m+k}(x, \mu_{m,k}(y, z)).$$

Let $\tau : G_n \times G_m \to G_m \times G_n$ be the flip, and

$$\mu_{m,n} \circ \tau : G_n \times G_m \longrightarrow G_{n+m}$$

be the opposite multiplication. Homotopy commutativity of the monoid M means that the triangle



commutes up to homotopy, but not necessarily preserving basepoints. At the level of fundamental groups this means that the corresponding diagram commutes up to conjugation, so there exist elements $c_{n,m} \in G_{n+m}$ such that $c_{n,m}^{-1} \cdot \mu_{n,m}(-) \cdot c_{n,m} = \mu_{m,n} \circ \tau(-)$.

Let G_∞ be the direct limit of the direct system of groups

$$G_0 \stackrel{\mu_{0,1}(-,e)}{\longrightarrow} G_1 \stackrel{\mu_{1,1}(-,e)}{\longrightarrow} G_2 \longrightarrow \cdots .$$

This is the fundamental group of the mapping telescope

$$M_\infty := \mathrm{hocolim}(M \stackrel{\cdot m_0}{\to} M \stackrel{\cdot m_0}{\to} M \stackrel{\cdot m_0}{\to} \cdots)$$

based at the unit $1_M \in M \subset M_\infty$.

Proposition 3.1. — *The derived subgroup of G_∞ is perfect.*

Proof. — Let $a, b \in G_n$ and consider $[a, b] \in G'_\infty$. Write $a \otimes b$ for $\mu_{n,m}(a, b)$ when $a \in G_n$ and $b \in G_m$, for ease of notation, and e_n for the unit of G_n.

In the direct limit we identify a with $a \otimes e_n$ and b with $b \otimes e_n$, and we have

$$b \otimes e_n = c_{n,n}^{-1}(e_n \otimes b)c_{n,n}$$

so $b \otimes e_n = [c_{n,n}^{-1}, (e_n \otimes b)] \cdot (e_n \otimes b)$. Thus $[a, b]$ is identified with

$$[a \otimes e_n, b \otimes e_n] = [a \otimes e_n, [c_{n,n}^{-1}, (e_n \otimes b)](e_n \otimes b)]$$

and because $e_n \otimes b$ commutes with $a \otimes e_n$ this simplifies to

$$[a \otimes e_n, [c_{n,n}^{-1}, (e_n \otimes b)]].$$

We now identify this with

$$[a \otimes e_{3n}, b \otimes e_{3n}] = [a \otimes e_{3n}, [c_{n,n}^{-1}, (e_n \otimes b)] \otimes e_{2n}]$$

and note that $a \otimes e_{3n} = c_{2n,2n}^{-1}(e_{2n} \otimes a \otimes e_n)c_{2n,2n} = [c_{2n,2n}^{-1}, (e_{2n} \otimes a \otimes e_n)] \cdot (e_{2n} \otimes a \otimes e_n)$. Again, as $(e_{2n} \otimes a \otimes e_n)$ commutes with $[c_{n,n}^{-1}, (e_n \otimes b)] \otimes e_{2n}$ the whole thing becomes

$$[a \otimes e_{3n}, b \otimes e_{3n}] = [[c_{2n,2n}^{-1}, (e_{2n} \otimes a \otimes e_n)], [c_{n,n}^{-1}, (e_n \otimes b)] \otimes e_{2n}],$$

a commutator of commutators. □

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