A NOTE ON THE FAMILY SIGNATURE THEOREM

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1. Twisted signatures

Let H be a $\mathbb{Z}$-module and $\lambda : H \otimes H \to \mathbb{Z}$ be a ε-symmetric nondegenerate bilinear form, with $\varepsilon \in \{1, -1\}$. Let $O(H, \lambda) \leq GL(H)$ denote the subgroup of those automorphisms of H which preserve the form λ. There is a corresponding local system of abelian groups $\mathcal{H} \to BO(H, \lambda)$, equipped with an ε-symmetric nondegenerate bilinear form.

There are cohomology classes

$$\sigma_i \in H^i(BO(H, \lambda); \mathbb{Q})$$

in degrees $i \equiv 0 \mod 4$ if $\varepsilon = +1$ and $i \equiv 2 \mod 4$ if $\varepsilon = -1$ defined as follows. Rational cohomology is dual to rational framed bordism, and if $[f : M \to BO(H, \lambda), \xi]$ is a framed bordism class of degree i as above then we set

$$\langle \sigma_i, [f, \xi] \rangle := \mathrm{sign}(H^{i/2}(M; f^*\mathcal{H})).$$

Here the signature of $H^{i/2}(M; f^*\mathcal{H})$ is taken with respect to the bilinear form

$$H^{i/2}(M; f^*\mathcal{H}) \otimes H^{i/2}(M; f^*\mathcal{H}) \xrightarrow{\smile} H^i(M; f^*(\mathcal{H} \otimes \mathcal{H})) \xrightarrow{\lambda} H^i(M; \mathbb{Z}) = \mathbb{Z},$$

which is symmetric because the cup product and λ are either both symmetric or both antisymmetric. This expression is well-defined by the usual argument for cobordism-invariance of signatures.

Remark 1.1. As the signature is defined for symmetric bilinear forms on *real* vector spaces, the classes σ_i are in fact pulled back from classes $\sigma_i^{\mathbb{R}} \in H^i(O(H \otimes \mathbb{R}, \lambda); \mathbb{Q})$, where the latter denotes the cohomology of the *discrete* group. In Section 4 we will show how work of Atiyah implies they are even already defined in continuous group cohomology.

If (H', λ') is another ε-symmetric form, then from the description above it is clear that under the map

$$BO(H, \lambda) \times BO(H', \lambda') \longrightarrow BO(H \oplus H', \lambda \oplus \lambda')$$

the class $\sigma_i^{H \oplus H'}$ pulls back to $\sigma_i^H \otimes 1 + 1 \otimes \sigma_i^{H'}$. As a special case of this, it follows that the σ_i are compatible under stabilisation of ε-symmetric forms.

2. Signatures of Poincaré fibrations

Now suppose that $F^d \to E^{4k} \to B^{4k-d}$ is a fibration with Poincaré base and fibre (and hence Poincaré total space too [Got79]), which is oriented in the sense that $\pi_1(B)$ acts trivially on $H^d(F; \mathbb{Z})$. Let

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; \mathbb{Z})) \Rightarrow H^{p+q}(E; \mathbb{Z})$$

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denote the Serre spectral sequence for this fibration. Meyer [Mey72] has shown that there is an identity

$$\mathrm{sign}(E) = \mathrm{sign}(E_2^{*,*}) = \begin{cases} \mathrm{sign}(H^{(4k-d)/2}(B; \mathcal{H}^{d/2}(F;\mathbb{Z}))) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

Here the signature of $E_2^{*,*}$ is taken with respect to the form

$$E_2^{*,*} \otimes E_2^{*,*} \xrightarrow{\smile} E_2^{*,*} \longrightarrow \mathbb{Z}[4k-d, d]$$

where the latter map denotes the projection to $E_2^{4k-d,d} = H^{4k-d}(B; \mathcal{H}^d(F;\mathbb{Z})) = \mathbb{Z}$.

Remark 2.1. Meyer assumes various additional hypotheses, most notably that B and F are homology manifolds. This is because he wishes to allow (non locally constant) sheaf coefficients. For locally constant coefficients being Poincaré complexes suffices for his argument to go through.

3. The family signature theorem

The following in the main result we want to record.

Theorem 3.1. *Let $\pi : E \to |K|$ be an oriented topological block bundle with fibre F^d. If d is even let $H := H^{d/2}(F;\mathbb{Z})/tors$, $\lambda : H \otimes H \to \mathbb{Z}$ denote the intersection form, and let $f : B \to BO(H, \lambda)$ classify the local system $\mathcal{H}^{d/2}(F;\mathbb{Z})/tors$ over B. Then*

$$\kappa_{\mathcal{L}_i}(\pi) := \int_\pi \mathcal{L}_i(T_\pi E) = \begin{cases} f^*(\sigma_{4i-d}) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

Proof. As rational cohomology is dual to rational framed bordism, by naturality it is enough to consider the case where $|K| = B^{4i-d}$ is a stably framed smooth manifold of dimension $4i - d$.

By the discussion in [HLLRW17, Section 2] a topological block bundle has a stable vertical tangent microbundle $T_\pi E \to E$, and as B is a topological manifold then by [HLLRW17, Lemma 2.5.2] E is a topological manifold and its stable tangent microbundle satisfies

$$TE \cong_s \pi^*(TB) \oplus T_\pi E \cong_s \mathbb{R}^{2k-d} \oplus T_\pi E$$

so $\mathcal{L}_i(T_\pi E) = \mathcal{L}_i(TE)$ by stability of the L-classes, and hence

$$\int_B \int_\pi \mathcal{L}_i(T_\pi E) = \int_E \mathcal{L}_i(TE) = \mathrm{sign}(E).$$

A block bundle determines a fibration with homotopy equivalent fibre, total space, and base, so by Meyer's result we have

$$\mathrm{sign}(E) = \begin{cases} \mathrm{sign}(H^{(4k-d)/2}(B; \mathcal{H}^{d/2}(F;\mathbb{Z}))) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

and by definition we have

$$\mathrm{sign}(H^{(4k-d)/2}(B; \mathcal{H}^{d/2}(F;\mathbb{Z}))) = \int_B f^*\sigma_{4k-d}.$$

Putting these together gives the required identification. □

The key ingredient making this argument work is that a topological block bundle over a topological manifold has topological manifold total space, and the Hirzebruch signature theorem holds for topological manifolds (by definition of the topological Hirzebruch L-classes).

Corollary 3.2. *The natural maps*

$$\begin{aligned}\mathbb{Q}[\sigma_2,\sigma_6,\sigma_{10},\ldots] &\longrightarrow \lim_{g\to\infty} H^*(BSp_{2g}(\mathbb{Z});\mathbb{Q})\\ \mathbb{Q}[\sigma_4,\sigma_8,\sigma_{12},\ldots] &\longrightarrow \lim_{g\to\infty} H^*(BO_{g,g}(\mathbb{Z});\mathbb{Q})\end{aligned} \tag{3.1}$$

are isomorphisms.

Proof. By a theorem of Ebert [Ebe11] there are smooth oriented fibre bundles with d-dimensional fibres, d any even number, having the classes $\kappa_{\mathcal{L}_i}$ arbitrarily algebraically independent. This implies that the maps in question are injective. On the other hand, by a theorem of Borel [Bor74] the right- and left-hand sides are abstractly isomorphic, so the map is in fact an isomorphism. □

Remark 3.3. This argument shows that the maps (3.1) are injective without using anything from the theory of arithmetic groups. We have seen that the classes σ_i are primitive, so this gives a lower bound on the hermitian K-theory of $\mathbb{Z}$ logically separate from the work of Borel.

4. Discussion

4.1. **Relation to work of Atiyah.** If $F^d \to E^{4k} \xrightarrow{\pi} B^{4k-d}$ is a smooth oriented fibre bundle over a stably framed base, with d even, then Atiyah has shown [Ati69, p. 83], using the index theorem, that

$$\mathrm{sign}(E) = \int_B \mathrm{ch}(\xi_\pi)$$

for $\xi_\pi \in K^0(B)$ given as follows:

(i) If $d/2$ is odd then the anti-symmetric form on the local system $\mathcal{H}^{d/2}(F;\mathbb{R}) \to B$ determines, after choosing a Riemannian metric on this vector bundle, a complex structure, and
$$\xi_\pi := \mathcal{H}^{d/2}(F;\mathbb{R}) - \overline{\mathcal{H}^{d/2}(F;\mathbb{R})} \in K^0(B).$$

(ii) If $d/2$ is even then the symmetric form on the local system $\mathcal{H}^{d/2}(F;\mathbb{R}) \to B$ determines, after choosing a Riemannian metric on this vector bundle, a decomposition $\mathcal{H}^{d/2}(F;\mathbb{R})^+ \oplus \mathcal{H}^{d/2}(F;\mathbb{R})^-$ into positive- and negative-definite subspaces, and
$$\xi_\pi := (\mathcal{H}^{d/2}(F;\mathbb{R})^+ - \mathcal{H}^{d/2}(F;\mathbb{R})^-) \otimes \mathbb{C} \in K^0(B).$$

In both cases these virtual vector bundles are defined over $BO(H,\lambda)$, giving a class $\xi_H \in K^0(BO(H,\lambda))$. In fact they are already defined over $BO(H\otimes\mathbb{R},\lambda)$, where $O(H\otimes\mathbb{R},\lambda)$ denotes the *topological* group of automorphisms of the form $(H\otimes\mathbb{R},\lambda)$.

Theorem 4.1. *We have*

$$ch_{4k-d}(\xi_H) = \sigma_{4k-d} \in H^{4k-d}(BO(H,\lambda);\mathbb{Q}).$$

Proof. Both terms are compatible under stabilisation, so it is enough to establish this identity in the stable range. As described above, by a theorem of Borel [Bor74] the maps (3.1) are isomorphisms, so $ch_{4k-d}(\xi_H)$ is equal to some polynomial $p(\sigma_1,\sigma_2,\ldots,\sigma_{4k-d})$ in the σ_j's. By Ebert's theorem [Ebe11], the polynomial $p(\sigma_1,\sigma_2,\ldots,\sigma_{4k-d}) - \sigma_{4k-d}$ is zero if and only if it pulls back to zero along $f : B^{4k-d} \to BO(H,\lambda)$ for every smooth fibre bundle $F^d \to E^{4k} \xrightarrow{\pi} B^{4k-d}$ over a stably framed base. But for such a fibre bundle we have

$$\int_B f^*(ch_{4k-d}(\xi_H) - \sigma_{4k-d}) = 0,$$

by the above discussion, as both $\int_B f^*(ch_{4k-d}(\xi_H))$ and $\int_B f^*(\sigma_{4k-d})$ calculate the signature of E. □

This shows that Atiyah's Chern character description of the signature is valid even for topological block bundles.

4.2. **Tautological relations.** The first step of the proof of the results of [GGRW17] is Theorem 2.1 of that papers, which says that for any smooth oriented fibre bundle $F^{2n} \to E \xrightarrow{\pi} B$ the classes $\kappa_{\mathcal{L}_i}(\pi) \in H^{4i-2n}(B;\mathbb{Q})$ are nilpotent. (In particular for the universal such bundle, so the degree of nilpotence is independent of the bundle.) This was deduced from Atiyah's theorem and from the fact that $O(H,\lambda)$, being an arithmetic group, has finite $\mathbb{Q}$-cohomological dimension.

By the discussion in this note, this conclusion holds not just for smooth fibre bundles but more generally for topological block bundles (and hence a fortiori for e.g. topological fibre bundles). It then follows from the argument of [GGRW17, Proposition 3.3 (i)] that the classes

$$\kappa_{p_I} \in H^*(B\widetilde{\mathrm{Homeo}}^+(W_g);\mathbb{Q})$$

are all nilpotent.

The other relations used in [GGRW17] come from Sections 5.4 and 5.5 of [Gri17] and these seem to really use the fibre bundle structure, as they involve describing e.g. the classifying space for fibre bundles with a section as the total space of the universal bundle, which is no longer true for block bundles.[1] It would be interesting to understand whether these relations also hold for block bundles.

4.3. **Fibrations with untrivialised boundaries.** Let W be a d-dimensional manifold with $\partial W = S^{d-1}$. Consider the data of an oriented relative fibration

$$(\pi, \partial\pi) : (E, \partial E) \longrightarrow B$$

with fibre $(W, \partial W)$. This has a fibre-integration, or Gysin, map

$$\int_\pi : H^i(E, \partial E) \longrightarrow H^{i-d}(B).$$

Generalising the discussion in Section 3.1 of [HLLRW17] to fibres which are manifolds with boundary defines a fibrewise Euler class $e^{fw}(\pi) \in H^d(E, \partial E;\mathbb{Q})$.

Suppose that the fibration is in addition equipped with a map $\tau : E \to BO$ and a section $s : B \to \partial E$, such that $\tau \circ s : B \to BO$ is the constant map to the basepoint. Given a polynomial $c = c(p_1, p_2, \ldots, p_i) \in H^*(BO;\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots]$ in the Pontrjagin classes, if $i > 0$ then $(e^{fw}(\pi))^i \cdot \tau^* c$ is a relative cohomology class and we can form the generalised Miller–Morita–Mumford class

$$\kappa_{e^i c}(\pi) = \int_\pi (e^{fw}(\pi))^i \cdot \tau^* c.$$

We wish to extend this to $i = 0$, and so make sense of the generalised Miller–Morita–Mumford class

$$\kappa_c(\pi) = \int_\pi \tau^* c.$$

This does not superficially make sense, as $\tau^* c \in H^*(E;\mathbb{Q})$ is an absolute, rather than relative, cohomology class and so cannot be fibre-integrated.

Lemma 4.2. *If $|c| > 0$ then the restriction of $\tau^* c$ to ∂E is zero.*

[1]The homotopy fibre of $B\widetilde{Homeo}^+(W_g, *) \to B\widetilde{Homeo}^+(W_g)$ is the space $\widetilde{Emb}(*, W_g)$ of block embeddings of a point, which has W_g as a retract but is larger.

Proof. The fibration $S^{d-1} \to \partial E \xrightarrow{\partial\pi} B$ has a section $s : B \to \partial E$, and so its Serre spectral sequence collapses and furthermore splits. Let $u \in H^{d-1}(\partial E;\mathbb{Q})$ be characterised by the properties of restricting to the generator of $H^{d-1}(S^{d-1};\mathbb{Q})$ and satisfying $s^*(u) = 0$. There is then an exact sequence

$$0 \longrightarrow H^{i-(d-1)}(B;\mathbb{Q}) \xrightarrow{u\cdot(\partial\pi)^* -} H^i(\partial E;\mathbb{Q}) \xrightarrow{s^*} H^i(B;\mathbb{Q}) \longrightarrow 0.$$

Now $s^*(\tau^*c)|_{\partial E} = 0$ so $(\tau^*c)|_{\partial E} = u \cdot (\partial E)^* x$ for some $x \in H^{|c|-d}(B;\mathbb{Q})$. But then we have

$$x = \int_{\partial\pi} u \cdot (\partial E)^* x = \int_{\partial\pi} (\tau^*c)|_{\partial E}$$

which vanishes by Stokes' theorem as $(\tau^*c)|_{\partial E}$ is the restriction of a cocycle on $E \supset \partial E$. Thus $x = 0$ and so $(\tau^*c)|_{\partial E} = 0$ as claimed. □

It follows that we may choose a lift $\overline{\tau^*c}$ of τ^*c to a relative cohomology class, which can be fibre-integrated. However, the integral so obtained is not well-defined, as for any $x \in H^{|c|-d}(B;\mathbb{Q})$ the class $\overline{\overline{\tau^*c}} = \delta^*(u\cdot(\partial\pi)^*x) + \overline{\tau^*c}$ is another a relative cohomology class, but

$$\int_\pi \overline{\overline{\tau^*c}} = x + \int_\pi \overline{\tau^*c}.$$

On the other hand, if $c' \in H^*(BO;\mathbb{Q})$ is another class, with lifts $\overline{\tau^*c'}$ and $\overline{\overline{\tau^*c'}} = \delta^*(u\cdot(\partial\pi)^*x') + \overline{\tau^*c'}$, then

$$\begin{aligned}\int_\pi \overline{\overline{\tau^*c}} \cdot \overline{\overline{\tau^*c'}} &= \int_\pi (\delta^*(u\cdot(\partial\pi)^*x) + \overline{\tau^*c})\cdot(\delta^*(u\cdot(\partial\pi)^*x') + \overline{\tau^*c'}) \\ &= \int_\pi \delta^*(u\cdot(\partial\pi)^*x)\cdot\overline{\tau^*c'} + \delta^*(u\cdot(\partial\pi)^*x')\cdot\overline{\tau^*c} + \overline{\tau^*c}\cdot\overline{\tau^*c'}\end{aligned}$$

as products $\delta^*(a)\cdot\delta^*(b)$ vanish, so

$$= \int_\pi \overline{\tau^*c}\cdot\overline{\tau^*c'} + \int_{\partial\pi} u\cdot(\partial\pi)^*x\cdot\overline{\tau^*c'}|_{\partial E} + u\cdot(\partial\pi)^*x'\cdot\overline{\tau^*c}|_{\partial E}$$

but by Lemma 4.2 again the classes $\overline{\tau^*c}|_{\partial E}$ and $\overline{\tau^*c'}|_{\partial E}$ vanish. Thus fibre integrals of non-trivial products of Pontrjagin classes are well-defined.

On the other hand we may write the ith Hirzebruch L-class as

$$\mathcal{L}_i = A_i p_i + D_i$$

where $A_i \in \mathbb{Q}^\times$ and D_i is a sum of non-trivial products of Pontrjagin classes. By the above discussion we may make sense of

$$\int_\pi \tau^* D_i \in H^{4i-d}(B;\mathbb{Q}).$$

On the other hand, if d is even then we have a map $f : B \to BO(H,\lambda)$ where $H = H_{d/2}(W)$ and λ is the intersection form, and so inspired by Theorem 3.1 we may define

$$\int_\pi \tau^*\mathcal{L}_i := \begin{cases} f^*(\sigma_{4i-d}) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd,}\end{cases}$$

and hence

$$\int_\pi \tau^* p_i := \frac{1}{A_i}\left(\int_\pi \tau^*\mathcal{L}_i - \int_\pi \tau^* D_i\right).$$

By this discussion we have defined $\kappa_{e^i p_I}(\pi) \in H^*(B;\mathbb{Q})$ for all $i \geq 0$ and all monomials p_I in Pontrjagin classes, in such a way that the conclusion of Theorem 3.1 holds.

This should be compared with the model for $B\widetilde{\mathrm{Diff}}_\partial(W)$ presented in Section 4 of [BM14] (more precisely, the model for $B\widetilde{\mathrm{Diff}}^J_\partial(W)$ obtained by combining

Corollaries 4.13 and 4.15). Berglund–Madsen's model classifies the kind of data introduced at the beginning of this section, and the discussion above shows (i) why this model has Miller–Morita–Mumford classes at all and (ii) why they satisfy the family signature theorem (namely, by definition).

The discussion in this section is closely related to Proposition 1.3.1 of [Wei15].

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