CORRIGENDUM TO: CONFIGURATION SPACES AS COMMUTATIVE MONOIDS

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Abstract. We correct the statement of Theorem 1.2 and its proof. In full generality that theorem is only valid rationally, but in some key cases the integral statement is still valid.

1. The issue

We refer throughout to the paper [RW24]. The first named author has observed that the statement of Theorem 1.2 cannot possibly be true, by direct calculation with $\mathbb{F}_p$-homology:

Counterexample. Let $M = \{\text{pt.}\}$ be the 1-point manifold, $\pi : L \to M$ be the trivial rank d vector bundle with $d \geq 3$ odd, and p be an odd prime. Writing $\mathrm{Sym}^k_*(-) = (-)^{\wedge k}_{\mathfrak{S}_k}$ for the symmetric powers in based spaces, and using that p is odd, one sees that $(L \oplus L)^+_{\mathfrak{S}_2} = \mathrm{Sym}^2_*(S^d)$ has trivial reduced $\mathbb{F}_p$-homology. The reduced homology of a based symmetric power depends only on the reduced homology of the input, by a theorem of Dold [Dol58, Theorem 7.2], so it follows that $\mathrm{Sym}^k_*([(L \oplus L)^+]_{\mathfrak{S}_2})$ also has trivial $\mathbb{F}_p$-homology for each $k > 0$, and therefore the map $\epsilon : \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) \to S^0[0]$ is a $\mathbb{F}_p$-homology isomorphism. If the square in [RW24, Theorem 1.2] were a homotopy pushout of commutative monoids, then it would follow that the map $\mathbf{Com}(L^+[1]) \to \mathbf{C}(\{\text{pt.}\}; L)$ is a $\mathbb{F}_p$-homology isomorphism too. But in grading p this map is $\mathrm{Sym}^p_*(S^d) \to \{\text{pt.}\}$, which is not a $\mathbb{F}_p$-homology isomorphism by e.g. [Nak61, Theorem 6.7]. □

The error is the claim in the proof of Lemma 3.2 that the first displayed square is a pushout in $\mathsf{Top}^{\mathbb{N}}_*$. This claim is true when $\pi : L \to M$ is the trivial rank 0 vector bundle, which is the case corresponding to Theorem 1.1: hence Theorem 1.1 seems to be correct as stated. But it is not true when the bundle $\pi : L \to M$ has strictly positive rank. (A further error is that $\mathbf{S}$ is not a subobject of $\mathbf{R}$ in this case.) The same issue arises in the setting treated in Appendix A; we comment on it later.

2. The correction

We will explain how to modify the argument to prove the following variant of (the incorrect) Theorem 1.2.

Theorem 1.2′. *There is a pushout square*

$$\begin{array}{ccc} \mathbf{Com}([(L \oplus L)^+]_{\mathfrak{S}_2}[2]) & \xrightarrow{\epsilon} & S^0[0] \\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow \\ \mathbf{Com}(L^+[1]) & \longrightarrow & \mathbf{C}(M; L) \end{array}$$

of unital commutative monoids in $\mathsf{Top}^{\mathbb{N}}_*$, where ϵ is the augmentation and Δ is induced by the diagonal inclusion $[(L \oplus L)^+]_{\mathfrak{S}_2} \to [L^+ \wedge L^+]_{\mathfrak{S}_2} = \mathbf{Com}(L^+[1])(2)$. The induced map from the homotopy pushout of commutative monoids

$$\mathbf{Com}(L^+[1]) \otimes^{\mathbb{L}}_{\mathbf{Com}([(L\oplus L)^+]_{\mathfrak{S}_2}[2])} S^0[0] \longrightarrow \mathbf{C}(M; L)$$

is a rational homology equivalence, and is an integral homology equivalence if L has rank at most 1.

The applications discussed in Section 2 of the published paper are all in rational (co)homology, and so follow from Theorem 1.2′ with no changes required. Taking L of rank (at most) 1 allows one to treat all M's except those which are non-orientable and even-dimensional, so even integrally most cases of interest are covered.

3. Proof of Theorem 1.2′

Let us first comment on how it can be that the first displayed square in the proof of Lemma 3.2 is a pushout in $\mathsf{Top}_*^{\mathbb{N}}$ when $L \to M$ is the trivial rank 0 vector bundle, but not more generally. For a vector bundle $L \to M$, we think of elements of $\mathbf{Com}(L^+[1])$ as (the point at ∞ together with) unordered configurations of points in M, possibly with repeats, where a point at location $m \in M$ is equipped with a label in the fibre L_m. When L has rank 0, so the label data is trivial, every element can be *canonically* written as $\mu + 2\lambda$, where the unordered tuple μ of points in M has no repeats. But when the rank of L is positive this is no longer possible: although the underlying points in M can be canonically partitioned in this way, if there is an odd number of labels at the same point of M then there is no way to decide which of these labels should be attributed to μ and which to 2λ.

We can define a variant $\mathbf{R}'$ of $\mathbf{R} = \mathbf{Com}(L^+[1])$ that precisely solves this problem: it consists of (the point at ∞ together with) unordered configurations of points in M, possibly with repeats, where a point at location $m \in M$ is equipped with a label ℓ in the fibre L_m, *and where whenever there is an odd number of labels at the same location in M then one of them is selected.* This is again a commutative monoid by superposition, where we 'deselect' points when necessary; we then let $\mathbf{S}' \subseteq \mathbf{R}'$ be the submonoid consisting of those tuples where each point in M carries an even number of labels. We topologise $\mathbf{R}'(n)$ as the one-point compactification of a space $\mathbf{R}'_o(n)$ given as the set described above, and with the quotient topology induced by the surjective map

$$\Phi : L^n \longrightarrow \mathbf{R}'_o(n)$$

which sends $(\ell_1, \ldots, \ell_n)$ to the unordered collection of these points where whenever there is an odd number of ℓ_i's lying over the same point of M, then the one with the smallest index i is the selected one. There is a filtration $\{F_p\mathbf{R}'\}_p$ by the subspaces $F_p\mathbf{R}'(n) \subset \mathbf{R}'(n)$ where there are $\leq p$ selected points (i.e. $\leq p$ points in M carry an odd number of labels), and $\mathbf{S}' = F_0\mathbf{R}'$. The unordered configuration space $C_n(M; L)$ is homeomorphic to the subspace of $\mathbf{R}'_o(n)$ consisting of those configurations where each point of M carries ≤ 1 label (which will necessarily be selected).

Lemma 3.2 is then replaced by the following.

Lemma 3.2′. $\mathbf{R}'$ *is a flat* $\mathbf{S}'$*-module, in the sense that* $\mathbf{R}' \otimes_{\mathbf{S}'} -$ *preserves weak equivalences between left* $\mathbf{S}'$*-modules whose underlying objects are well-based.*

Proof. By the proof of Lemma 3.2 it suffices to show that for each $p \geq 1$ the square

$$\begin{array}{ccc} F_{p-1}\mathbf{R}'(p)[p] \otimes \mathbf{S}' & \longrightarrow & F_{p-1}\mathbf{R}' \\ \downarrow & & \downarrow \\ \mathbf{R}'(p)[p] \otimes \mathbf{S}' & \longrightarrow & F_p\mathbf{R}' \end{array}$$

is a pushout in $\mathsf{Top}_*^{\mathbb{N}}$. This can be checked in each grading n.

We first claim that $F_{p-1}\mathbf{R}(n)$ and the image of $(\mathbf{R}'(p)[p] \otimes \mathbf{S}')(n) = \mathbf{R}'(p) \wedge \mathbf{S}'(n-p)$ cover $F_p\mathbf{R}'(n)$, i.e. that configurations having exactly p selected points come from $\mathbf{R}'(p) \wedge \mathbf{S}'(n-p)$. This is because such configurations can be written as the superposition of p points which are all selected (coming from $\mathbf{R}'(p)$) with a configuration having an even number of labels at each point of M (coming from $\mathbf{S}(n-p)$). We then claim that the preimage of $F_{p-1}\mathbf{R}'(n)$ in $\mathbf{R}'(p) \wedge \mathbf{S}'(n-p)$ is $F_{p-1}\mathbf{R}'(p) \wedge \mathbf{S}'(n-p)$. This is because superposition with a configuration which has an even number of labels at each point does not affect the collection of points having an odd number of labels. This proves that the square is a pushout of sets, i.e. the map from the pushout to $F_p\mathbf{R}'(n)$ is a continuous bijection. As the spaces involved are compact Hausdorff, it is a homeomorphism. □

Applying $\mathbf{R}' \otimes_{\mathbf{S}'} -$ to the weak equivalence $B(\mathbf{S}', \mathbf{S}', S^0[0]) \xrightarrow{\simeq} S^0[0]$, it follows that the map

$$B(\mathbf{R}', \mathbf{S}', S^0[0]) \longrightarrow \mathbf{R}' \otimes_{\mathbf{S}'} S^0[0]$$

is an equivalence. The right-hand term is $\mathbf{C}(M; L)$, just as in Lemma 3.4, as tensoring over $\mathbf{S}'$ with $S^0[0]$ identifies all configurations on $\mathbf{R}'$ having ≥ 2 labels over some point of M with the basepoint. To finish the argument, we will show that:

Proposition 3.6′. *The natural surjective maps*

$$\mathbf{Com}((L \oplus L)^+]_{\mathfrak{S}_2}[2]) \longrightarrow \mathbf{S}'$$
$$\mathbf{R}' \longrightarrow \mathbf{Com}(L^+[1])$$

are rational homology equivalences, and are integral homology equivalences if L has rank at most 1.

This gives a zig-zag of maps

$$\begin{array}{ccc} B(\mathbf{R}', \mathbf{Com}((L \oplus L)^+]_{\mathfrak{S}_2}[2]), S^0(0) & \longrightarrow & B(\mathbf{R}', \mathbf{S}', S^0[0]) \\ \downarrow & & \\ B(\mathbf{Com}(L^+[1]), \mathbf{Com}((L \oplus L)^+]_{\mathfrak{S}_2}[2]), S^0[0]) & & \end{array}$$

which are rational—or integral if L has rank at most 1—homology equivalences and so establishes Theorem 1.2′. The proof of Proposition 3.6′ uses the following elementary lemma about pointed symmetric powers.

Lemma 3.7′. *The natural maps*

$$\mathrm{Sym}_*^j(\mathrm{Sym}_*^2(S^d)) \to \mathrm{Sym}_*^{2j}(S^d), \textit{ and } S^d \wedge \mathrm{Sym}_*^{m-1}(S^d) \to \mathrm{Sym}_*^m(S^d) \textit{ with } m \neq 2,$$

are rational homology equivalences, and are integral homology equivalences if $d = 1$.

Proof. For a pointed space X there are canonical isomorphisms

$$\widetilde{H}_*(\mathrm{Sym}_*^k(X); \mathbb{Q}) \cong \mathrm{Sym}^k(\widetilde{H}_*(X; \mathbb{Q})).$$

If d is odd this shows that $\mathrm{Sym}_*^k(S^d)$ has trivial rational homology for all $k \geq 2$; if d is even then it shows that the natural quotient map $(S^d)^{\wedge k} \to \mathrm{Sym}_*^k(S^d)$ is a rational homology isomorphism. This proves the lemma in either case.

If $d = 1$ then $\mathrm{Sym}_*^k(S^1)$ is contractible for all $k \geq 2$. This follows by recognising it as the 1-point compactification of $\mathrm{Sym}^k(\mathbb{R})$, then using that points in $\mathbb{R}$ are canonically ordered to obtain a homeomorphism $\mathrm{Sym}^k(\mathbb{R}) \cong \mathbb{R} \times [0, \infty)^{k-1}$, whose 1-point compactification is $S^1 \wedge [0, \infty]^{\wedge k-1} \simeq *$. Using this we see that the source and target of all maps in question are contractible, except in the case $m = 1$, where the map in question is the identity map of S^1. □

Proof of Proposition 3.6′. Throughout, we take cohomology with rational or integral coefficients as the case may be. The source and target (with the points at ∞ removed) of each map admit filtrations F'_k by closed subspaces by asking that there be $\leq k$ distinct points in M which carry labels; the maps respect these filtrations. Let

$S_k(-)$ denote the kth strata of these filtrations, i.e. the loci where there are precisely k points in M with labels. It suffices to show that the induced maps on strata induce isomorphisms on compactly-supported cohomology; the map of spectral sequences in compactly supported cohomology for these filtrations then gives the desired conclusion.

The source and target of

$$S_k\mathbf{Com}((L \oplus L)^+]_{\mathfrak{S}_2}[2])(n) \longrightarrow S_k\mathbf{S}'(n) \tag{3.1}$$

fibre over the unordered configuration space $C_k(M)$ by recording the underlying points in M. On fibres over $\{m_1, \dots, m_k\} \in C_k(M)$, this map takes the form

$$\coprod_{n=2n_1+\cdots+2n_k} \prod_{i=1}^{k} \mathrm{Sym}^{n_i}(\mathrm{Sym}^2(L_{m_i})) \longrightarrow \coprod_{n=2n_1+\cdots+2n_k} \prod_{i=1}^{k} \mathrm{Sym}^{2n_i}(L_{m_i}),$$

This induces an isomorphism on compactly-supported cohomology by Lemma 3.7′. The Leray–Serre spectral sequence in compactly-supported cohomology then implies that (3.1) is an isomorphism on compactly-supported cohomology, as required.

The source and target of

$$S_k\mathbf{R}' \longrightarrow S_k\mathbf{Com}(L^+[1]) \tag{3.2}$$

similarly fibre over $C_k(M)$, and on fibres over $\{m_1, \dots, m_k\} \in C_k(M)$, this map is

$$\coprod_{n=n_1+\cdots+n_k} \prod_{i=1}^{k} \widetilde{\mathrm{Sym}}^{n_i}(L_{m_i}) \longrightarrow \coprod_{n=n_1+\cdots+n_k} \prod_{i=1}^{k} \mathrm{Sym}^{n_i}(L_{m_i})$$

where

$$\widetilde{\mathrm{Sym}}^{m}(V) := \begin{cases} \mathrm{Sym}^m(V) & m \text{ even} \\ V \times \mathrm{Sym}^{m-1}(V) & m \text{ odd.} \end{cases}$$

This again induces an isomorphism on compactly-supported cohomology by Lemma 3.7′. The Leray–Serre spectral sequence in compactly-supported cohomology then implies that (3.2) is an isomorphism on compactly-supported cohomology, as required. □

Remark 3.1. The integral result in Theorem 1.2′ is sharp in that the analogue for $\mathrm{rank}(L) = 2$ does not hold. For example, suppose the conclusion of that Theorem holds for $M = \mathbb{R}^2$ with the trivial rank 2 vector bundle $L \to M$. Fix an odd prime p and take $\mathbb{F}_p$ coefficients in the following.

One computes that $\widetilde{H}_*([(L \oplus L)^+]_{\mathfrak{S}_2}) = \Sigma^6\mathbb{F}_p$ and $\widetilde{H}_*(L^+) = \Sigma^4\mathbb{F}_p$; the latter implies that $\widetilde{H}_*(\mathrm{Sym}^2_*(L^+)) \simeq \Sigma^8\mathbb{F}_p$ since $2 \in \mathbb{F}_p^\times$. The map $\widetilde{C}_*([(L \oplus L)^+]_{\mathfrak{S}_2}) \to \widetilde{C}_*(\mathrm{Sym}^2_*(L^+))$ therefore factors (up to homotopy) through 0. Following Dold [Dol58, Section 7], we may identify $\widetilde{C}_*(\mathbf{Com}(L^+[1]))$ with the free simplicial commutative $\mathbb{F}_p$-algebra $\mathbb{L}\mathrm{Sym}^*(\widetilde{C}_*(L^+)[1])$, and hence see that the induced map $\widetilde{C}_*(\mathbf{R}) \to \widetilde{C}_*(\mathbf{S})$ factors (up to homotopy of maps of simplicial commutative $\mathbb{F}_p$-algebras) through $\mathbb{F}_p = \mathbb{L}\mathrm{Sym}^*(0)$. It follows that we may write

$$\widetilde{C}_*(\mathbf{C}(M; L); \mathbb{F}_p) \simeq (\mathbb{F}_p \otimes^{\mathbb{L}}_{\mathbb{L}\mathrm{Sym}^*(\Sigma^6\mathbb{F}_p[2])} \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{L}\mathrm{Sym}^*(\Sigma^4\mathbb{F}_p[1]),$$

where we have implicitly chosen formality equivalences for the chains. In particular, $\mathbb{L}\mathrm{Sym}^*(\Sigma^4\mathbb{F}_p[1])$ is a homotopy retract of $\widetilde{C}_*(\mathbf{C}(M; L); \mathbb{F}_p)$. But one computes that $H_{2p+2}(\mathbb{L}\mathrm{Sym}^p(\Sigma^4\mathbb{F}_p)) \neq 0$ (for example by considering the homology of $\mathrm{Sym}^p_*(S^4)$ [Nak61, Theorem 6.7], or applying [Bou, Theorem 8.11]), so also

$$H^{2p-2}(C_p(\mathbb{R}^2); \mathbb{F}_p) \simeq H_{2p+2}(C_p(\mathbb{R}^2; L)^+; \mathbb{F}_p) \neq 0,$$

where the first equivalence is an application of Poincaré duality (as in [RW24, §2.1]). This contradicts the theorem of Arnol'd that $H^i(C_n(\mathbb{R}^2); \mathbb{Z}) = 0$ for $i \geq n$ [Arn70].

4. Analogous corrections to Appendix A

The same oversight arises in the claimed equivalence (A.2). Again, it is still true rationally and is true integrally if L has rank 0 (and if L has rank 1 and $m = 1$, as we will see below). If L has rank $d > 0$ then one should instead make a variant $\mathbf{R}'$ of the $\mathbb{N}^m$-graded topological monoid $\mathbf{R} := \mathbf{Com}(\bigvee_{i=1}^m L^+[1_i])$. To do so, say that a collection of coloured labels in L_m is *k-uniform* if there is an ℓ such that there are precisely $k \cdot \ell$ labels of each colour. (This generalises having an even number of labels over each point.) Define $\mathbf{R}'$ to be (the point at ∞ together with) unordered configurations of points in M, possibly with repeats, where each point is equipped with one of m colours as well as a label in the fibre of $L \to M$ over it, *and where a subset of the labels is selected, such that for each point of M there is a colour having $< k$ selected labels, and the non-selected labels over each point of M are k-uniform.* (This somewhat complicated formulation is just the analogue of what we did previously.) There is a filtration $F_p\mathbf{R}'$ by those configurations having $\leq p$ points with selected labels, and one sets $\mathbf{S}' := F_0\mathbf{R}'$ and shows that $\mathbf{R}'$ is a flat left $\mathbf{S}'$-module just as in Lemma 3.2′. We deduce that

$$B(\mathbf{R}', \mathbf{S}', S^0[0, \ldots, 0]) \longrightarrow \mathbf{R}' \otimes_{\mathbf{S}'} S^0[0, \ldots, 0] = \mathbf{Z}^{m,k}(M; L)$$

is an equivalence. This is the correct integral version of (A.2).

To deduce something closer to (A.2) itself we use the natural surjections

$$\mathbf{Com}([(L^{\oplus mk})^+]_{\mathfrak{S}_m^k}[k, \ldots, k]) \longrightarrow \mathbf{S}'$$

$$\mathbf{R}' \longrightarrow \mathbf{Com}(\bigvee_{i=1}^m L^+[1_i])$$

which the argument of Proposition 3.6′ shows are rational homology equivalences, using that the natural maps

$$\mathrm{Sym}_*^\ell(\mathrm{Sym}_*^k((S^d)^{\wedge m})) \to \mathrm{Sym}_*^{\ell k}((S^d)^{\wedge m}), \text{ and } \mathrm{Sym}_*^i(S^d) \wedge \mathrm{Sym}_*^{\ell k}(S^d) \to \mathrm{Sym}_*^{i+\ell k}(S^d)$$

are rational homology equivalences for $0 \leq i < k$. (They are also integral homology equivalences if $d = 0$ or if $d = 1$ and $m = 1$ as in Lemma 3.7′.) This shows that (A.2) is a rational homology equivalence, and an integral homology equivalence if L has rank 0, or has rank 1 and $m = 1$.

The applications discussed in Section A.2 are in rational homology, so are unchanged; the applications discussed in Section A.3 treat the case L has rank 0, so are unchanged.

References

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