ON DIFFEOMORPHISMS ACTING ON ALMOST COMPLEX STRUCTURES

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Abstract. We construct some diffeomorphisms (by twisting around certain handles in a manifold) which do not preserve any stable almost complex structure on the ambient manifold and yet act trivially on homology. In particular, they are not isotopic to symplectomorphisms.

This note arose from a question asked of the author by Ivan Smith, regarding the existence of diffeomorphisms of symplectic manifolds which are homologically trivial and yet do not preserve the symplectic structure, even up to isotopy. We shall give a construction of a family of such diffeomorphisms, and show that they cannot preserve a symplectic structure by showing that they act without fixed points on the set of stable almost-complex structures on the manifold.

The method was inspired by the paper of Hajduk–Tralle [4], who attempt to use diffeomorphisms supported in a disc to achieve this goal. Unfortunately, [4] contains an error, and as explained in [2] any diffeomorphism supported in a disc *does* preserve any almost-complex structure.

1. **Construction.** Let us write $S^n = D^n_+ \cup D^n_-$ for the sphere decomposed into upper and lower hemispheres, and define

$$W^{n+m} = S^n \times D^m_- \cup_{D^n_- \times D^m_-} D^n_- \times S^m.$$

After rounding corners, this manifold is diffeomorphic to the complement of an open disc in $S^n \times S^m$. In particular there is an identification of ∂W with S^{n+m-1} such that $W \cup_{\partial W} D^{n+m} \approx S^n \times S^m$. Suppose we are given a smooth map $\phi : S^n \to SO(m)$ which sends an open neighbourhood of D^n_- to the identity matrix. Then we obtain a diffeomorphism

$$(x, y) \mapsto (x, \phi(x) \cdot y) : S^n \times D^m_- \longrightarrow S^n \times D^m_-$$

which is the identity on $D^n_- \times D^m_-$, and so extends to a diffeomorphism $\bar{\phi}$ of W.

Lemma 1.1. *The diffeomorphism $\bar{\phi}$ acts trivially on the homology of W.*

Proof. The fundamental classes of the spheres $S^n \times \{0\}$ and $\{0\} \times S^m$ generate the homology of W, but these spheres are fixed pointwise by $\bar{\phi}$. □

Lemma 1.2. *The restriction of $\bar{\phi}$ to ∂W is (non-canonically) isotopic to the identity, as long as $n + m \geq 5$.*

Proof. Let $f = \bar{\phi}|_{\partial W}$. Then $\bar{\phi}$ extends to a diffeomorphism

$$\bar{\phi} \cup \mathrm{Id} : W \cup_{\mathrm{Id}} D^{n+m} \longrightarrow W \cup_f D^{n+m},$$

showing that $S^n \times S^m \approx (S^n \times S^m)\#\Sigma_f$, where Σ_f is the exotic $(n+m)$-sphere obtained by the clutching construction using f. However, by [5, Theorem A], the inertia group of $S^n \times S^m$ is trivial and so $\Sigma_f \approx S^{n+m}$. By the theorems of Smale [6] and Cerf [1], identifying the group of homotopy $(n+m)$-spheres with the group

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of isotopy classes of diffeomorphisms of S^{n+m-1} for $n+m \geq 5$, it follows that f is isotopic to the identity. $\square$

Choosing an isotopy from $\bar{\phi}|_{\partial W}$ to the identity and using isotopy extension, we may change $\bar{\phi}$ by an isotopy to a diffeomorphism $\tilde{\phi}$ which is the identity on the boundary. Furthermore, we may assume that $\tilde{\phi}$ is equal to $\bar{\phi}$ near $(S^n \times \{0\}) \cup (\{0\} \times S^m)$. (We leave it to the interested reader to convince themselves that all such $\tilde{\phi}$ so obtained differ by diffeomorphisms supported in a disc.)

2. **Stable framings and stable almost complex structures.** Let M be a d-dimensional manifold.

Definition 2.1. A *stable framing* ξ on M is the data of a bundle isomorphism $\xi : \epsilon^{k+d}_{\mathbb{R}} \to \epsilon^k_{\mathbb{R}} \oplus TM$ for some $k \gg 0$, where we allow ourselves to change ξ by a bundle homotopy and to replace it by $\epsilon^1_{\mathbb{R}} \oplus \xi$.

A *stable almost complex structure* $\zeta = (E, b)$ on M is a pair of a n-dimensional complex vector bundle $E \to M$ and an isomorphism of real vector bundles $b : E \to \epsilon^{2n-d}_{\mathbb{R}} \oplus TW$. We allow ourselves to replace E by an isomorphic bundle, change b by a bundle homotopy, and replace (E, b) by $(\epsilon^1_{\mathbb{C}} \oplus E, \epsilon^2_{\mathbb{R}} \oplus b)$.

To each stable framing ξ there is associated a stable almost complex structure $\xi_{\mathbb{C}}$, called the complexification, obtained by increasing k until $(k + d)$ is even, taking $E = \epsilon^{(k+d)/2}_{\mathbb{C}}$, and letting b be given by ξ and the identity homeomorphism $\epsilon^{(k+d)/2}_{\mathbb{C}} = \epsilon^{k+d}_{\mathbb{R}}$.

We let $SF(M)$ denote the set of equivalence classes of stable framings of M, and $SC(M)$ denote the set of equivalence classes of stable almost complex structures. By standard bundle theory, they are in bijection with the set of homotopy classes of lifts of a (stable) Gauss map $\tau_M : M \to BO$ along

$$EO \longrightarrow BO \quad \text{or} \quad BU \longrightarrow BO$$

respectively. As these are both fibrations of infinite loop spaces, with fibres O and O/U respectively, there are free and transitive actions (i.e. torsors)

$$[M, O] \circlearrowright SF(M) \quad \text{and} \quad [M, O/U] \circlearrowright SC(M).$$

The actions may be described as follows. Firstly, an element $x \in [M, O]$ is represented by a self-isomorphism $c : \epsilon^k_{\mathbb{R}} \to \epsilon^k_{\mathbb{R}}$ for some $k \gg 0$, and $x \cdot \xi$ is represented by $\epsilon^k_{\mathbb{R}} \oplus \xi$ precomposed with the automorphism $c \oplus \xi$. Secondly, an element $y \in [M, O/U]$ is represented by a pair (F, c) where $F \to M$ is a e-dimensional complex vector bundle and $c : F \to \epsilon^{2e}_{\mathbb{R}}$ is a trivialisation as a real vector bundle. If $\zeta = (E, b) \in SC(W; \zeta|_{\partial W})$ then $y \cdot \zeta$ is given by

$$(F \oplus E, F \oplus E \xrightarrow{c \oplus b} \epsilon^{2e}_{\mathbb{R}} \oplus \epsilon^{2n-d}_{\mathbb{R}} \oplus TW).$$

Complexifying gives a function $SF(M) \to SC(M)$, and this is equivariant with respect to the homomorphism $q \circ - : [M, O] \to [M, O/U]$ induced by $q : O \to O/U$.

3. **The action of diffeomorphisms.** If $f : M \to M$ is a diffeomorphism and ξ is a stable framing, we obtain a new stable framing $f^*\xi$ by

$$\mathbb{R}^{k+d} \times M \xrightarrow{\mathrm{Id} \times f} \mathbb{R}^{k+d} \times M \xrightarrow{\xi} \epsilon^k \oplus TM \xrightarrow{\epsilon^k \oplus Df^{-1}} \epsilon^k \oplus TM.$$

This satisfies $g^*(f^*\xi) = (f \circ g)^*\xi$, so defines a right action of the mapping class group $\Gamma(M)$ on $SF(M)$.

Similarly, if $\zeta = (E, b)$ is a stable almost complex structure, we obtain a new one $f^*\zeta = (f^*E, f^*b)$ where f^*b is given by

$$f^*E \xrightarrow{\bar{f}} E \xrightarrow{b} \epsilon^{2n-d}_{\mathbb{R}} \oplus TM \xrightarrow{\epsilon^{2n-d}_{\mathbb{R}} \oplus Df^{-1}} \epsilon^{2n-d}_{\mathbb{R}} \oplus TM.$$

Again, we obtain a right action of $\Gamma(M)$ on $SC(M)$.

3.1. *The action on stable framings of* W. The manifold $W^{n+m} \approx S^n \times S^m \setminus D^{n+m}$ may be (unstably) framed, so let $\xi^0 : \epsilon_{\mathbb{R}}^{n+m} \to TW$ be a choice of framing, that is, a bundle isomorphism. Then for each point $x \in W$ we obtain a linear map

$$\mathbb{R}^{n+m} \times \{x\} \overset{\xi^0_{f(x)}}{\longrightarrow} T_{f(x)}W \overset{Df^{-1}}{\longrightarrow} T_x W \overset{(\xi^0_x)^{-1}}{\longrightarrow} \mathbb{R}^{n+m} \times \{x\},$$

which determines a continuous map $\delta(f) : W \to GL_{n+m}(\mathbb{R})$, which sends ∂W to the identity. By definition, $f^*\xi^0$ is obtained from ξ^0 by precomposing it with the automorphism of $\epsilon_{\mathbb{R}}^{n+m}$ given by $\delta(f)$.

As $SF(W)$ is a $[W, O]$-torsor, for each f and ξ there is an element $\Delta(f, \xi) \in [W, O]$ such that $f^*\xi = \Delta(f, \xi) \cdot \xi$. By definition of the torsor structure this satisfies

$$f^*(x \cdot \xi) = (x \circ f) \cdot f^*\xi = (x \circ f) \cdot \Delta(f, \xi) \cdot \xi = (x \circ f) \cdot \Delta(f, \xi) \cdot x^{-1} \cdot (x \cdot \xi)$$

and so $\Delta(f, x \cdot \xi) = (x \circ f) \cdot \Delta(f, \xi) \cdot x^{-1}$.

The class $\Delta(f, \xi^0)$ is the stabilisation of $\delta(f)$. Thus f fixes the framing $x \cdot \xi^0$ if and only if $(x \circ f) \cdot \Delta(f, \xi^0) = x \in [W, O]$.

3.2. *The action on stable almost complex structures of* W. Similarly to the last section, as $SC(W)$ is a $[W, O/U]$-torsor, to each f and ζ there is a $\Delta^{\mathbb{C}}(f, \zeta) \in [W, O/U]$ such that $f^*\zeta = \Delta^{\mathbb{C}}(f, \zeta) \cdot \zeta$. This satisfies $\Delta^{\mathbb{C}}(f, y \cdot \zeta) = (y \circ f) \cdot \Delta^{\mathbb{C}}(f, \zeta) \cdot y^{-1}$, as above. In particular, if ξ^0 is an unstable framing as above and $\zeta^0 = \xi^0_{\mathbb{C}}$ is its complexification, then

$$\Delta^{\mathbb{C}}(f, \zeta^0) = q \circ \Delta(f, \xi^0).$$

Thus for $y \in [W, O/U]$, f fixes the almost complex structure $y \cdot \zeta^0$ if and only if $(y \circ f) \cdot q(\Delta(f, \xi^0)) = y \in [W, O/U]$.

3.3. *The action of* $\tilde{\phi}$. The diffeomorphism $\tilde{\phi}$ of W constructed in §1 fixes pointwise the subset $(S^n \times \{0\}) \cup (\{0\} \times S^m)$. As W deformation retracts to this subset, $\tilde{\phi}$ is homotopic to the identity. Thus for any $y \in [W, O/U]$ the maps $y \circ f$ and y are homotopic. Thus $\tilde{\phi}$ fixes the stable almost complex structure $y \cdot \zeta^0$ if and only if $q(\Delta(f, \xi^0)) = 0 \in [W, O/U]$. Note that this condition is independent of y: $\tilde{\phi}$ either fixes all almost complex structures on W or fixes none of them.

A necessary condition for $\tilde{\phi}$ to fix the complex structures on W is that the composition

$$S^n \times \{0\} \longrightarrow W \overset{q(\Delta(\tilde{\phi}, \xi^0))}{\longrightarrow} O/U$$

is nullhomotopic. In other words, as $\Delta(\tilde{\phi}, \xi^0)$ is the stabilisation of $\delta(\tilde{\phi})$, the composition

$$S^n \overset{\phi}{\longrightarrow} SO(m) \longrightarrow O \overset{q}{\longrightarrow} O/U$$

must be nullhomotopic.

Theorem 3.1. *If either*

(i) $n \equiv 7 \mod 8$, $n \geq 23$ *and* $m \geq \frac{n+3}{2}$,
(ii) $n = 15$ *and* $m \geq 13$,
(iii) $n = 7$ *and* $m \geq 8$,
(iv) $n \equiv 0 \mod 8$, $n \geq 8$ *and* $m \geq 6$,

then there exists a $\phi \in \pi_n(SO(m))$ *such that any* $\tilde{\phi}$ *acts without fixed points on* $SC(W)$.

Proof. By the discussion above, it is sufficient to find a $\phi \in \pi_n(SO(m))$ so that

$$S^n \xrightarrow{\phi} SO(m) \longrightarrow SO \xrightarrow{q} SO/U$$

is not nullhomotopic. The map q is non-trivial on π_n only when n is congruent to 7 or 0 modulo 8, in which case it is a surjection to $\mathbb{Z}/2$. Thus it is enough to show that under the conditions listed above, the map

$$\pi_n(SO(m)) \longrightarrow \pi_n(SO)$$

is surjective. For (i) and (iv) this follows from [3, Theorem 1.1], and for (ii) and (iii) this follows from [3, Proposition 2.1]. □

4. **Implanting the diffeomorphism.** Given a codimension zero embedding $W \hookrightarrow M^{n+m}$, we may extend the diffeomorphism $\tilde{\phi}$ of W by the identity on $M \setminus \mathrm{Int}(W)$ to obtain a diffeomorphism φ of M. If ζ is a stable almost complex structure on M, then the difference between ζ and $\varphi^*\zeta$ is given by a $z \in [M, O/U]$ whose restriction to W is $\Delta^{\mathbb{C}}(\tilde{\phi}, y \cdot \zeta^0)$ for some $y \in [W, O/U]$. As $\tilde{\phi}$ is homotopic to the identity, this is the same as $\Delta^{\mathbb{C}}(\tilde{\phi}, \zeta^0) = q(\Delta(\tilde{\phi}, \xi^0))$, which is non-trivial whenever ϕ is obtained from Theorem 3.1, so $\zeta \neq \varphi^*\zeta$. Hence φ acts without fixed points on the stable almost complex structures on M.

Corollary 4.1. *Let (M^{2k}, ω) be a symplectic manifold, n and m integers satisfying the assumptions of Theorem 3.1 such that $n + m = 2k$, and $e : W^{n+m} \hookrightarrow M$ a codimension zero embedding. Then $\varphi^*\omega$ is not isotopic to ω, where φ is the diffeomorphism obtained by extending $\tilde{\phi}$ by the identity (which acts trivially on the homology of M).*

Remark 4.2. Ivan Smith has pointed out that a compact Kähler manifold (M^{2k}, ω), or more generally a strong Lefschetz manifold, can never contain an embedded copy of W^{n+m} with $n + m = 2k$ and $n \neq m$: if $n < m$, say, then by the hard Lefschetz theorem the map

$$- \cdot \omega^{\frac{m-n}{2}} : H^n_{dR}(M) \longrightarrow H^m_{dR}(M)$$

is an isomorphism. But ω becomes trivial on W^{n+m}, so this map cannot hit the element of $H^m_{dR}(M)$ Poincaré dual to $S^n \times \{0\} \subset W \subset M$. Thus in this case we must settle for W^{k+k}-summands.

As an example, let $X^{2k} \subset \mathbb{CP}^{k+r}$ be a smooth codimension r complete intersection, with $2k \geq 6$. If either

(i) k is odd and $b_k(X) \geq 6$, or
(ii) k is even and $\min(b_k^+(X), b_k^-(X)) \geq 3$,

then classical techniques of surgery theory show that X has a $S^k \times S^k$ connect-summand. Thus if in addition $k \equiv 7 \mod 8$ and $k \geq 15$, or $k \equiv 0 \mod 8$ and $k \geq 8$, then there is a diffeomorphism of X which acts freely on the set of isotopy classes of symplectic forms on X.

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