ALGEBRAIC TOPOLOGY (PART III) **MICHELMAS 2009**

EXAMPLE SHEET 2

PART A

1. (*Reduced homology*) Let X be a space, and recall that $C_0(X) = \langle e_x \,|\, x \in X \rangle$. Define a homomorphism $\phi : C_0(X) \to \mathbb{Z}$ by setting $\phi(e_x) = 1$ and extending linearly. Show that ϕ descends to a well-defined map $\phi_* : H_0(X) \to \mathbb{Z}$. The *reduced homology* $\widetilde{H}_n(X)$ is defined to be $H_n(X)$ for $n > 0$ and $\ker \phi_*$ for $n = 0$. Show that $\widetilde{H}_*(X) \simeq H_*(X, x)$ for any $x \in X$.

2. If X is a space, the *cone* on X is defined to be $CX = X \times [0,1]/(X \times 1)$. If $f : X \to Y$ is a map, the *mapping cone* of f is $C(f) = Y \cup_F CX$, where $F : X \times 0 \to Y$ is given by $F(x, 0) = f(x)$.

   (a) Suppose $A \subset X$, and let $\iota : A \to X$ be the injection. Show that $\widetilde{H}_*(C(\iota)) \simeq H_*(X, A)$.

   (b) The *suspension* of X is defined to be $\Sigma X = CX/(X \times 0)$. Show that $\widetilde{H}_*(\Sigma X) \cong \widetilde{H}_{*-1}(X)$.

3. Consider the cell structure on S^n which has two cells of each dimension between 0 and n, corresponding to the northern and southern hemispheres of S^k. Write out its cellular chain complex and verify that it has the correct homology. What does this have to do with $\mathbb{RP}^n$?

4. Suppose G_n ($n > 0$) is a finitely generated abelian group, and that all but finitely many G_n are 0. Construct a finite cell complex X with $H_n(X) \simeq G_n$ for all $n > 0$.

5. Recall that $A \in GL_2(\mathbb{Z})$ defines a map $A : T^2 \to T^2$. Let X_A be the space obtained by starting with $X \times [0,1]$ and identifying $(x, 0)$ with $(A(x), 1)$. Compute $H_*(X_A)$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

6. Show there is a connected closed 4-manifold with Euler characteristic n for all $n \in \mathbb{Z}$. Is analogous statement for 2-manifolds true?

7. Show that $H^1(X)$ is free for any space X.

8. Define a natural map $H^n(X; G) \to Hom(H_n(X; G), G)$, and show that if G is a field this map is an isomorphism. For $G = \mathbb{Z}$, give an example where it is not.

PART B

1. (*The Puppe sequence*) Suppose $A \subset X$ and that $i : A \to X$ is the inclusion. Show that there is a sequence of maps

$$A \xrightarrow{i} X \xrightarrow{j} C(i) \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma(i)} \Sigma X \xrightarrow{\Sigma(j)} C(\Sigma(i)) \to \dots$$

such that if we apply H_n to this sequence, we get the long exact sequence of the pair (X, A). What is δ? From this, deduce that the boundary map in the long exact sequence is *natural*, in the sense that if $f : (X, A) \to (Y, B)$, there is an induced map of long exact sequences so that all squares commute.

2. Let $X = S^1 \vee S^2$. Give an example of a map $f : X \to X$ which is not homotopic to the identity on X, but for which $f_* = 1_{H_*(X)}$. (Hint: lift the map of S^2 to the universal cover.) Is $C(f)$ contractible?

3. Suppose $x \in H_n(X)$, where X is an arbitary topological space. Show that there is a finite cell complex A and a map $f : A \to X$ so that $x \in \operatorname{im} f_*$.

4. Let $R = \mathbb{C}[x]/(x^3)$, and for $i = 1, 2$, let M_i be the R-module $\mathbb{C}[x]/(x^i)$. Find a free resolution of M_1 and use it to compute $\operatorname{Tor}_*^R(M_1, M_1)$ and $\operatorname{Tor}_*^R(M_1, M_2)$.

5. Suppose X is a finite cell complex, and that $p : \tilde{X} \to X$ is the universal covering map. Let $G = \pi_1(X)$, so that G acts on $\tilde{X}$ as the group of deck transformations.

   (a) If we give $\tilde{X}$ the cell structure lifted from X, show that $C_*^{cell}(\tilde{X})$ can be viewed as a chain complex over the group ring $R = \mathbb{Z}[G]$.

   (b) If $X = T^2$, describe $C_*^{cell}(\tilde{X})$ as a complex over the group ring $\mathbb{Z}[\mathbb{Z}^2] \simeq \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$.

   (c) Suppose that $\tilde{X}$ is contractible, and that $\pi : X' \to X$ is a normal covering map with deck group K. Show that $H_*(X) = \operatorname{Tor}_*^R(\mathbb{Z}, \mathbb{Z}[K])$. (First decide how R acts on $\mathbb{Z}$ and $\mathbb{Z}[K]$.)

6. With notation as in the previous problem, suppose that H is an abelian group, and that $\phi : G \to Aut(H)$ is a homomorphism. Explain how ϕ can be used to make H into a module over R. The homology of the chain complex $C_*(\tilde{X}) \otimes_R G$ is called the homology of X with coefficients twisted by ϕ (written $H_*(X; \phi)$).

   (a) Let $H = \mathbb{C}$. What are the possible values of $H_*(T^2; \phi)$ as ϕ varies over homomorphisms $\pi_1(T^2) \to \mathbb{C}^*$?

   (b) Suppose $f : X \to X$ is a homeomorphism, and let $Y = X \times [0, 1]/\sim$, where $(x, 1) \sim (f(x), 0)$. Show that $H_*(Y) \simeq H_*(S^1; \phi)$, where $\phi : \mathbb{Z} \to Aut(H_*(X))$ is defined by $\phi(1) = f_*$.

J.Rasmussen@dpmms.cam.ac.uk