ALGEBRAIC TOPOLOGY (PART III) **MICHELMAS 2009**

EXAMPLE SHEET 1

PART A

0. If you've never met homotopy equivalence before, spend an afternoon thinking about homotopy type of common objects. Which of the following are homotopy equivalent to (a) a point (b) a circle (d) a wedge of circles? A fork; a cheese grater; a bicycle tire; a bicycle inner tube; a bicycle frame; your desk chair. (Warning: results may vary depending on your furniture and on how close you look.)

1. If $X_1 \sim X_2$ and $Y_1 \sim Y_2$, show there is a bijection between the sets $[X_1, Y_1]$ and $[X_2, Y_2]$.

2. Let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{R}$ be given by $\sigma_1(x) = x, \sigma_2(x) = 1-x$. Identifying $[0,1]$ with Δ gives a cycle $e_{\sigma_1} + e_{\sigma_2}$ in $C_1(\mathbb{R})$. Give an explicit example of $x \in C_2(\mathbb{R})$ with $dx = e_{\sigma_1} + e_{\sigma_2}$.

3. Given that G is abelian, what can you say about the isomorphism type of G if

   (a) $0 \to \mathbb{Z}^n \to G \to \mathbb{Z}^m \to 0$ is exact?

   (b) $0 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/4 \to 0$ is exact?

4. Let X be the genus 2 surface shown in Figure 1.

   (a) Use the Mayer Vietoris sequence to compute $H_*(X)$. (Hint: divide X along A.)

   (b) Let A, B and C be curves as shown in the figure. What are $H_*(X - A), H_*(X - B)$ and $H_*(X - C)$.

   (c) Use the exact sequence of a pair to compute $H_*(X, A), H_*(X, B)$ and $H_*(X, C)$.

   (d) (If you know something about π_1.) Is X/A homotopy equivalent to X/C?

5. Suppose $f : T^2 \to T^2$ is a homeomorphism. Show that $f_* : H_1(T^2) \to H_1(T^2)$ defines an element of $GL(2, \mathbb{Z})$, and that any element of $GL(2, \mathbb{Z})$ can be realized by a homeomorphism of T^2. (Hint: consider linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$).

6. Let X be the quotient of $S^n \times [0,1]$ obtained by identifying $(x, 0)$ and $(-x, 1)$. Use the Mayer-Vietoris sequence to compute $H_*(X)$. (Your answer should depend on n.)

7. Write $X = S^1 \times D^2$, and let $A \subset X$ be the boundary torus. ($A = S^1 \times S^1 \subset S^1 \times D^2$.) Compute $H_*(X, A)$.

8. Show S^{n+m+1} can be decomposed as the union of $S^n \times D^{m+1}$ and $D^{n+1} \times S^m$ along their common boundary $S^n \times S^m$. Use this to compute $H_*(S^n \times S^m)$. (Hint: $S^n = \{(x_1, \ldots, x_{n+1}, 0, 0, \ldots, 0) \mid \sum_{i=0}^{n+1} x_i^2 = 1\} \subset S^{n+m+1}$.)

PART B

1. Show that $GL(n, \mathbb{R})$ is homotopy equivalent to $O(n)$. (Hint: Gramm-Schmidt.) What is the analogous statement for $GL(n, \mathbb{C})$?

2. Let $C_1 = (C_*, d)$ be a chain complex defined over a field k, and let $C_2 = (H_*(C), 0)$ be the complex whose underlying group is $H_*(C)$, with trivial differential. Show that C_1 and C_2 are chain homotopy equivalent. Give an example to show that if we replace k by $\mathbb{Z}$, the analogous statement is false.

3. Suppose (C_*, d_C) and (D_*, d_D) are two chain complexes defined over a ring R. Let $F_i = Hom_i(C, D)$ be the set of R-linear maps $f : C_* \to D_*$ which raise the grading by i — *i.e.* $f(C_j) \subset D_{i+j}$ for all j. Define $d_F : F_* \to F_{*-1}$ by $d_F(f) = f \circ d_C + (-1)^i d_D \circ f$. Show that
   (a) (F_*, d_F) is a chain complex
   (b) $f \in F_0$ is a cycle if and only if it defines a chain map from C_* to D_*.
   (c) if f and g are two cycles in F_0, $[f] = [g]$ if and only if f is chain homotopic to g.

4. Given that $H_*(X)$ is a free abelian group, use the Mayer-Vietoris sequence to show that $H_*(X \times S^1) \cong H_*(X) \oplus H_{*-1}(X)$. (Actually, this is true even if $H_*(X)$ is not free.) Use this to compute $H_*(T^n)$.

5. If $f : S^n \to S^n$ has the property that $f(x) \neq -x$ for all $x \in S^n$, show that $f \sim 1_{S^n}$. Similarly, if $f(x) \neq x$ for all x, show that f is homotopic to the antipodal map.

6. A *vector field* on S^n is a continuous map which assigns to each $x \in S^n$ a vector in the tangent space to S^n at v. Equivalently, we can think of it as a continuous map $f : S^n \to \mathbb{R}^{n+1}$ with the property that $\langle x, f(x) \rangle = 0$ for all $x \in S^n$. Use the previous problem to show that when n is even, there is no nonvanishing vector field on S^{2k}. Give an example of an nonvanishing vector field on S^{2k+1}.

7. Let $i : S^1 \times D^2 \to S^3$ be an embedding, and let U be the interior of its image. Use the Mayer-Vietoris sequence to compute $H_*(S^3 - U)$ and show that it is independent of i. (Is the homotopy type of $S^3 - U$ independent of i?)

J.Rasmussen@dpmms.cam.ac.uk