ALGEBRAIC TOPOLOGY (PART III) MICHAELMAS 2017

EXAMPLE SHEET 1

Problems 6, 8, and 10 will be marked if handed in by 5pm on October 24.

1. If $X_1 \sim X_2$ and $Y_1 \sim Y_2$, show there is a bijection between the sets $[X_1, Y_1]$ and $[X_2, Y_2]$.

2. Let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{R}$ be given by $\sigma_1(x) = x, \sigma_2(x) = 1 - x$. Identifying $[0,1]$ with Δ^1 gives a cycle $e_{\sigma_1} + e_{\sigma_2}$ in $C_1(\mathbb{R})$. Find an $x \in C_2(\mathbb{R})$ with $dx = e_{\sigma_1} + e_{\sigma_2}$.

3. (*Cancellation*) Suppose (C, d) is a chain complex, that $C_n = C'_n \oplus A$, $C_{n-1} = C'_{n-1} \oplus A$, and that the component of d_n mapping A to A is the identity map. Show that (C, d) is chain homotopy equivalent to (C', d'), where $C'_i = C_i$ for $i \neq n, n-1$, $d'_i = d_i$ for $i \neq n-1, n, n+1$, and d'_{n+1} is the composition of d_{n+1} with the projection onto C'_n. (Hint: use $d^2 = 0$ to determine d'_n.)

4. What are the possible isomorphism types of the abelian group G in the following exact sequences?

$$0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0 \qquad 0 \to \mathbb{Z} \to G \to \mathbb{Z}/4 \to 0$$
$$0 \to \mathbb{Z}/4 \to G \to \mathbb{Z} \to 0 \qquad 0 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/4 \to 0$$

5. * If $f : (C, d) \to (C', d')$ is a chain map, the *mapping cone* of f is the chain complex $(M(f), d_f)$ whose underlying group is given by $M(f)_n = C_{n-1} \oplus C'_n$, and whose differential is given by

$$(d_f)_n = \begin{pmatrix} d_{n-1} & 0 \\ (-1)^n f_{n-1} & d'_n \end{pmatrix}.$$

Show that $(M(f), d_f)$ is a chain complex, and that if $f \sim g$, then $M(f) \sim M(g)$. By considering an appropriate mapping cone, give a proof of the Five Lemma. (Hint: use Exercise 3.) If C and C' are free finitely generated chain complexes over $\mathbb{Z}$, prove that $H_*(M(f)) = 0$ if and only if f is a chain homotopy equivalence.

6. Let X be the genus 2 surface shown in Figure 1.

   (a) Use the Mayer-Vietoris sequence to compute $H_*(X)$. (Hint: divide X along A.)

   (b) Let A, B and C be curves as shown in the figure. What are $H_*(X - A), H_*(X - B)$ and $H_*(X - C)$?

   (c) Use the exact sequence of a pair to compute $H_*(X, A), H_*(X, B)$ and $H_*(X, C)$.

7. Let $i : S^1 \times D^2 \to S^3$ be an embedding, and let U be the interior of its image. Use the Mayer-Vietoris sequence to compute $H_*(S^3 - U)$. Show that $H_1(S^3 - U)$ is generated by $i_*([p \times S^1])$, where p is a point in S^1.

8. Show S^{n+m+1} can be decomposed as the union of $S^n \times D^{m+1}$ and $D^{n+1} \times S^m$ along their common boundary $S^n \times S^m$. Compute $H_*(S^n \times S^m)$ and $H_*(D^{n+1} \times S^m, S^n \times S^m)$.

9. Suppose $f : T^2 \to T^2$ is a homeomorphism. Show that $f_* : H_1(T^2) \to H_1(T^2)$ defines an element of $GL(2, \mathbb{Z})$, and that any element of $GL(2, \mathbb{Z})$ can be realized by a homeomorphism of T^2.

10. If $f : X \to X$ is a homeomorphism, let Y be the quotient of $X \times [0,1]$ obtained by identifying $(x, 0)$ and $(f(x), 1)$. Show there is a long exact sequence

$$\longrightarrow H_{n+1}(Y) \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \longrightarrow H_n(Y) \longrightarrow$$

(Hint: write Y as the union of $X \times [0, 1/2]$ and $X \times [1/2]$, use the Mayer-Vietoris sequence, and cancel.) Compute $H_*(Y)$ when $X = S^n$ and f is the antipodal map; when $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and f is multiplication by $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

11. If $H_*(X)$ is a free abelian group, show that $H_*(X \times S^1) \cong H_*(X) \oplus H_{*-1}(X)$. (In fact, this is true even if $H_*(X)$ is not free.) Compute $H_*(T^n)$.

12. * Let $L = L_1 \coprod L_2 \subset S^3$ be the union of two copies of S^1 embedded as shown in Figure 2. (We view S^3 as the one-point compactification of $\mathbb{R}^3$.) Let $\nu(L)$ be an open tubular neighborhood of L. (That is, an embedding of two copies of $\operatorname{int} D^2 \times S^1$ that restricts to the embedding shown on $0 \times S^1$.) Let $Y = S^3 - \nu(L)$.

    (a) Compute $H_*(Y)$, and show that $H_1(Y)$ is generated by the curves $[m_1]$ and $[m_2]$ shown in the figure.

    (b) The boundary ∂Y is a disjoint union of two tori T_1 and T_2. If $i : T_1 \to Y$ is the inclusion, compute $i_* : H_*(T_1) \to H_*(Y)$. (For H_1, express your answer in terms of $[m_1]$ and $[m_2]$.)

    (c) Use the long exact sequence of a pair to compute $H_*(Y, \partial Y)$. Sketch surfaces with boundary which generate $H_2(Y, \partial Y)$.

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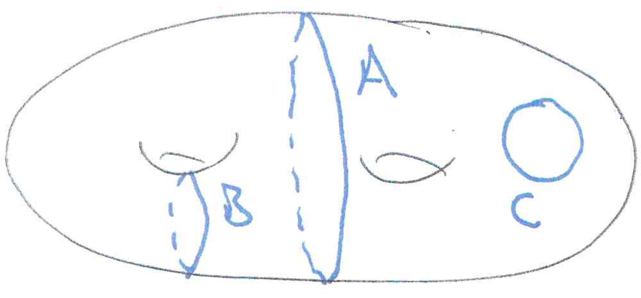

Figure 1

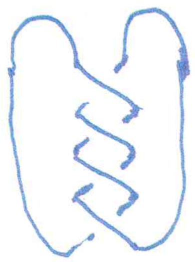

Figure 2