The Proof of the Subdivision Lemma

Recall that if $\mathcal{U} = \{U_\alpha\}$ is a open cover of X, $C_*^{\mathcal{U}}(X) = \langle \sigma \mid \operatorname{im} \sigma \subset U_\alpha \text{ for some } \alpha \rangle$ is a subcomplex of $C_*(X)$. The goal of this note is to prove

Theorem 1 (The Subdivision Lemma)**.** *Let $\mathcal{U}$ be an open cover of X. If $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ is the inclusion, the induced map $i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.*

The proof of this theorem is similar in structure to the proof that if $f \sim g$, $f_\# \sim g_\#$. There are five steps:

- Define a chain map $B_n : S_*(\Delta^n) \to C_*(\Delta^n)$. B is called *barycentric subdivision.* It is defined by the requirements that
  (1) $B_0(f_0) = \mathrm{id}_{\Delta^0}$.
  (2) If $|I| < n$, $B_n(f_I) = F_{I\#} \circ B_{|I|}(\mathbf{f}_{|I|})$, where $F_I : \Delta^{|I|} \to \Delta^n$ is the face map.
  (3) $B_n(\mathbf{f}_n)$ is the cone on $B_n(d(\mathbf{f}_n))$.
- Let $\varphi_n : S_*(\Delta^n) \to C_*(\Delta^n)$ be the map induced by id_{Δ^n}. We show that $B_n \sim \varphi_n$ by a chain homotopy T_n.
- Next, we check that B_n and T_n are *natural* in the sense that if $F_I : \Delta^k \to \Delta^n$ is a face map, then $B_n \circ \phi_I = F_{I\#} \circ B_k$ and $T_n \circ \phi_I = F_{I\#} \circ T_k$.
- We define a map $B : C_*(X) \to C_*(X)$ by $B(\sigma) = \sigma_\#(B_n(\mathbf{f}_n))$ for $\sigma : \Delta^n \to X$. Naturality of B_n and T_n implies that B is a chain map and $B \sim \mathrm{id}_{C_*(X)}$.
- Suppose that $\mathcal{U}$ is an open cover of X. If $x \in C_*(X)$, we show there is some $r > 0$ so that $B^r(x) \in C_*^{\mathcal{U}}(X)$. Using this, we complete the proof.

Preliminaries

We start by fixing some notation. If $I = \{i_0, i_1 \ldots, i_k\} \subset \{0, \ldots, n\}$, we let $|I| = k$ (not $k+1$, as I used in class!) We define $F_I : \Delta^{|I|} \to \Delta^n$ be the corresponding face map. We have chain maps $\varphi_n : S_*(\Delta^k) \to C_*(\Delta^k)$ given by $\varphi_n(f_I) = F_I$, and $\phi_I : S_*(\Delta^k) \to S_*(\Delta^n)$ given by $\phi_I(f_J) = f_{I\circ J} := f_{i_{j_0}\ldots i_{j_l}}$, where $l = |J|$. Then we have $F_{I\circ J} = F_I \circ F_J$, which implies $\varphi_n \circ \phi_I = F_{I\#} \circ \varphi_k$. Finally, we define $\mathbf{f}_n := f_{01\ldots n} \in S_n(\Delta^n)$ to be the top dimensional face of Δ^n.

Next we discuss cones.

Definition 2. *If X is a space, the cone on X is $CX = X \times [0,1]/X \times 0$.*

A map $f : X \to Y$ induces a map $Cf : CX \to CY$ given by $Cf(x,t) = (f(x), t)$. The cone $C\Delta^{n-1}$ can be identified with Δ^n by the map ψ which sends $((x_0, \ldots, x_{n-1}, t)$ to $(1-t, tx_0, tx_1, \ldots, tx_{n-1})$. Thus $\sigma : \Delta^{n-1} \to X$ induces a map $c\sigma : \Delta^n \to CX$ given by $c\sigma = C\sigma \circ \psi^{-1}$. Hence we have a map $c : C_*(X) \to C_{*+1}(CX)$ given by $c(\sigma) = c\sigma$. If $i : X \to CX$ is the map given by $i(x) = (x, 1)$, it follows easily from the definition that

$$dc(\sigma) = i_\#(\sigma) - c(d\sigma).$$

Let $\pi : C\Delta_n \to \Delta_n$ be the map given by $\pi(\mathbf{v}, t) = t\mathbf{v} + (1-t)\mathbf{b}$, where $\mathbf{b} = \frac{1}{n+1}(1, \ldots, 1)$ is the barycenter of Δ_n, and define $\beta = \pi_\# \circ c : C_*(\Delta_n) \to C_{*+1}(\Delta^n)$. Since $\pi \circ i = \mathrm{id}_{\Delta^n}$, we have

$$d\beta(\sigma) = \sigma - \beta(d\sigma).$$

Barycentric Subdivision

We define a chain map $B_n : S_*(\Delta^n) \to C_*(\Delta_n)$ inductively. First, $B_0 : S_0(\Delta^0) \to C_0(\Delta^0)$ is uniquely defined by the requirement that $B_0(f_0) = \mathrm{id}_{\Delta^0}$. In general, if I is a proper subset of $\{0, \ldots, n\}$, we define

$$B_n(f_I) = F_{I\#}(B_{|I|}(\mathbf{f}_{|I|}))$$

Finally, we define

$$B_n(\mathbf{f}_n) = \beta(B_n(d\mathbf{f}_n)) \in C_n(\Delta_n).$$

Observe that all the singular simplices appearing in the image of B_n are given by affine linear maps.

Lemma 3. *B_n is a chain map.*

Proof. This is proved by induction on n. The case $n = 0$ is trivial. Given that B_k is a chain map for $k < n$, it follows from the definition that B_n is a chain map when restricted to $S_*(\Delta^n)$ where $* < n$. Thus the only thing to check is that $B_n(d\mathbf{f}_n) = dB_n(\mathbf{f}_n)$. We compute

$$\begin{aligned} dB_n(\mathbf{f}_n) &= d\beta(B_n(d\mathbf{f}_n)) \\ &= B_n(d\mathbf{f}_n) - \beta(dB_n(d\mathbf{f}_n)) \\ &= B_n(d\mathbf{f}_n) - \beta(B_n(d^2\mathbf{f}_n)) \\ &= B_n(d\mathbf{f}_n). \end{aligned}$$

We have used the fact that the statement holds in gradings $< n$ in passing from the second to the third lines. □

The Chain Homotopy

Next, we want to define a chain homotopy $T_n : S_*(\Delta_n) \to C_{*+1}(\Delta^n)$. As with the chain map B_n, we define T_n inductively. First, let T_0 be the zero map. Next, if I is a proper subset of $0, \dots, n$, define

$$T_n(f_I) = F_{I\#}(T_{|I|}(\mathbf{f}_{|I|})).$$

Finally, we define

$$T_n(\mathbf{f}_n) = \beta(B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n) - T_n(d\mathbf{f}_n))$$

Lemma 4. $dT_n + T_n d = B_n - \varphi_n$.

Proof. This is proved by induction on n. The case $n = 0$ is easily verified, since $T_0 = 0$ and $B_0 = \varphi_0$. Suppose the result holds for all $k < n$. As in the case of B_n, we need only verify the identity when both sides are applied to $\mathbf{f}_n$; the other cases follow from the induction hypothesis. For $\mathbf{f}_n$, we compute

$$\begin{aligned} dT_n(\mathbf{f}_n) &= d\beta(B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n) - T_n(d\mathbf{f}_n)) \\ &= B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n) - T_n(d\mathbf{f}_n) - \beta(B_n(d\mathbf{f}_n) - \varphi_n(d\mathbf{f}_n) - dT_n(d\mathbf{f}_n)) \\ &= B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n) - T_n(d\mathbf{f}_n) - \beta(T_n(d^2\mathbf{f}_n)) \\ &= B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n) - T_n(d\mathbf{f}_n) \end{aligned}$$

where we have used the fact that the identity holds for f_I with $|I| < n$ in going from the second to the third line. So $dT_n(\mathbf{f}_n) + T_n(d\mathbf{f}_n) = B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n)$ as desired.

□

Naturality

Lemma 5. *If $F_I : \Delta^k \to \Delta^n$ is a face map, then $B_n \circ \phi_I = F_{I\#} \circ B_k$ and $T_n \circ \phi_I = F_{I\#} \circ T_k$.*

Proof. We have

$$B_n(\phi_I(f_J)) = B_n(f_{I\circ J}) = (F_{I\circ J})_\#(B_{|J|}(\mathbf{f}_{|J|})) = F_{I\#}(F_{J\#}(B_{|J|}(\mathbf{f}_{|J|}))) = F_{I\#}(B_k(e_J)).$$

The proof of the second statement is identical, but with B's replaced by T's. □

Subdivision on X

If X is a space, define $B : C_*(X) \to C_*(X)$ by $B(\sigma) = \sigma_\#(B_n(\mathbf{f}_n))$ for $\sigma : \Delta^n \to X$. It is clear from the definition that if $g : X \to Y$, $B(g_\#(\sigma)) = g_\#(B(\sigma))$

Lemma 6. *B is a chain map.*

Proof. We compute

$$\begin{aligned} B(d\sigma) &= \sum(-1)^j B(\sigma \circ F_{\hat{j}}) \\ &= \sum(-1)^j \sigma_\#(F_{\hat{j}\#}(B_{n-1}(\mathbf{f}_{n-1}))) \\ &= \sum(-1)^j \sigma_\#(B_n(\phi_{\hat{j}}(\mathbf{f}_{n-1}))) \quad \text{(by Lemma \textbf{??})} \\ &= \sigma_\#(B_n(d\mathbf{f}_n)) \\ &= \sigma_\#(dB_n(\mathbf{f}_n)) \quad (B_n \text{ is a chain map}) \\ &= dB(\sigma) \end{aligned}$$

□

Lemma 7. *B is chain homotopic to* $\mathrm{id}_{C_*(X)}$.

Proof. Let us define $T : C_*(X) \to C_{*+1}(X)$ by $T(\sigma) = \sigma_\#(T_n(\mathbf{f}_n))$. As in the previous lemma, we compute

$$\begin{aligned} T(d\sigma) &= \sum(-1)^j T(\sigma \circ F_{\hat{j}}) \\ &= \sum(-1)^j \sigma_\#(F_{\hat{j}\#}(T_{n-1}(\mathbf{f}_{n-1}))) \\ &= \sum(-1)^j \sigma_\#(T_n(\phi_{\hat{j}}(\mathbf{f}_{n-1}))) \quad \text{(by Lemma \textbf{??})} \\ &= \sigma_\#(T_n(d\mathbf{f}_n)). \end{aligned}$$

Somewhat more easily, we have $dT(\sigma) = \sigma_\#(dT_n(\mathbf{f}_n))$, so

$$dT(\sigma) + Td(\sigma) = \sigma_\#(dT_n(\mathbf{f}_n) + T_n(d\mathbf{f}_n)) = \sigma_\#(B_n(\mathbf{f}_n) - \varphi_n(\mathbf{f}_n)) = B(\sigma) - \sigma.$$

□

Completing the proof

Let $\mathbf{F}_n = \varphi_n(\mathbf{f}_n) \in C_n(\Delta^n)$ be the singular simplex corresponding to the map id_{Δ^n}. If $\sigma : \Delta^n \to X$, then $\sigma = \sigma_\#(\mathbf{F}_n)$, so $B^r(\sigma) = \sigma_\#(B^r(\mathbf{F}_n))$. The simplices appearing in $B^r(\mathbf{F}_n)$ are all affine linear simplices obtained by iteratively applying barycentric subdivision to Δ^n.

Lemma 8. *If Δ is an affine linear simplex of dimension n, and Δ' is a simplex obtained by applying barycentric subdivision to Δ, then* $\mathrm{diam}(\Delta') \le \frac{n}{n+1}\mathrm{diam}(\Delta)$.

Proof. Let $\mathbf{v}_0, \ldots, \mathbf{v}_n$ be the vertices of Δ, so $d = \mathrm{diam}(\Delta) = \max \|\mathbf{v}_i - \mathbf{v}_j\|$. We induct on n. Suppose $\mathbf{v}, \mathbf{v}'$ are two vertices of Δ'. If $\mathbf{v}, \mathbf{v}'$ lie in a k-dimensional proper face Δ_I of Δ, they are vertices of a simplex appearing in the barycentric subdivision of Δ_I. By induction we have $\|\mathbf{v} - \mathbf{v}'\| \le \frac{k}{k+1}\mathrm{diam}(\Delta_I) \le \frac{n}{n+1}d$. So it suffices to consider the case where $\mathbf{v} = \frac{1}{n+1}(\mathbf{v}_0 + \ldots + \mathbf{v}_n)$ is the barycenter. Without loss of generality, we may assume the other vertex is of the form $\frac{1}{k+1}(\mathbf{v}_0 + \ldots + \mathbf{v}_k)$ for some k. Then

$$\mathbf{v}' - \mathbf{v} = \frac{1}{(n+1)(k+1)}\big[(n-k)\mathbf{v}_0 + \ldots + (n-k)\mathbf{v}_k - (k+1)\mathbf{v}_{k+1} \ldots - (k+1)\mathbf{v}_n\big].$$

The sum in the parentheses on the RHS can be rearranged into a sum of $(n-k)(k+1)$ terms of the form $\mathbf{v}_i - \mathbf{v}_j$, so

$$\|\mathbf{v}' - \mathbf{v}\| \le \frac{n-k}{n+1}\max\|\mathbf{v}_i - \mathbf{v}_j\| = \frac{n-k}{n+1}d \le \frac{n}{n+1}d.$$

□

Corollary 9. *If we normalize Δ^n to have diameter 1, then every simplex appearing in $B^r(\mathbf{F}_n)$ has diameter less than or equal to* $\left(\dfrac{n}{n+1}\right)^r$.

We will use the following standard fact about metric spaces:

Lemma 10. *If $\{U_i\}$ is an open cover of a compact metric space X, then there is some $\epsilon > 0$ so that any $A \subset X$ with diameter $< \epsilon$ is contained in some U_i.*

Proposition 11. *Let $\mathcal{U}$ be an open cover of X. If $x \in C_*(X)$, there is some $r > 0$ so that $B^r(x) \in C_*^{\mathcal{U}}(X)$.*

Proof. Since any $x \in C_*(X)$ is a finite linear combination of singular simplices, it suffices to prove the claim in the case where $x = \sigma$ for some $\sigma : \Delta^k \to X$. Let $V_i = \sigma^{-1}(U_i)$. Then the V_i are an open cover of Δ^n, so we can apply Lemma **??** to find an $\epsilon > 0$ such that any $A \subset X$ with diameter $< \epsilon$ is contained in some V_i. By Corollary **??**, we can choose r so that the diameter of any simplex in $B^r(\mathbf{F}_n)$ is $< \epsilon$. Thus, any simplex appearing in $B^r(\mathbf{F}_n)$ is contained in some V_i. Since $B^r(\sigma) = \sigma_\#(B^r(\mathbf{F}_n))$, every singular simplex appearing in $B^r(\sigma)$ is contained in some U_i. □

Proof of Theorem 1. We first show the map $i_* : H_*^{\mathcal{U}}(X) \to H_*(X)$ is surjective. Given $[x] \in H_*(X)$, we apply Proposition **??** to see that $B^r(x) \in C_*^{\mathcal{U}}(X)$ for some r. $B^r \sim \mathrm{id}_{C_*(X)}$, so $[x] = [B^r(x)]$. It follows that i_* is surjective.

Next, we show that i_* is injective. If $[x] \in H_*^{\mathcal{U}}(X)$ and $x = dy$ for some $y \in C_*(X)$, then we can find r so that $B^r(y) \in C_*^{\mathcal{U}}(X)$. Since B is a chain map, $B^r(x) = B^r(dy) = dB^r(y)$, so $[x] = [B^r(x)] = 0$ in $H_*^{\mathcal{U}}(X)$. □