Spherical/Projective Geometry

Spherical Lines

- A line is the shortest path between two points.
- Plane separation: the complement of a line is a disconnected topogical space.
- There is a unique line passing through two distinct, non-antipodal points.
- Two distinct lines intersect in two points.
- Given a point $\mathbf{x}$ and a line L not containing $\mathbf{x}$, there is a line passing through $\mathbf{x}$ and perpendicular to L.

Projective Lines

- A line is the shortest path between two points.
- The complement of a line is connected
- There is a unique line passing through two distinct, points.
- Two distinct lines intersect in exactly one point.
- Given a point $\mathbf{x}$ and a line L not containing $\mathbf{x}$, there is a line passing through $\mathbf{x}$ and perpendicular to L.

Circles

- A line and a circle which is distinct from it intersect in at most two points.
- Two distinct circles intersect in at most two points.
- The perimeter of a circle of radius R is $2\pi \sin R$.

Isometries

- If F_1 and F_2 are orthogonal frames, there is a unique isometry taking F_1 to F_2.
- An isometry which fixes three non-colinear points is the identity.
- Any isometry can be written as the composition of ≤ 3 reflections.

Triangles

- The sum of the interior angles in ΔABC is $\pi + \text{Area}(ABC)$.
- If A_1, A_2, A_3 and A'_1, A'_2, A'_3 are two sets of non-colinear points with $d(A_i, A_j) = d(A'_i, A'_j)$, then there is a unique $\phi \in \text{Isom}(\mathbb{R}^2)$ with $\phi(A_i) = A'_i$.
- If A_1, A_2, A_3 and A'_1, A'_2, A'_3 are two sets of non-colinear points with $d(A_1, A_j) = d(A'_1, A'_j)$ and $\angle A_2A_1A_3 = \angle A'_2A'_1A'_3$ then there is a unique $\phi \in \text{Isom}(\mathbb{R}^2)$ with $\phi(A_i) = A'_i$.

Trigonometry

If $\triangle ABC$ has sides a, b, c and opposite angles α, β, γ, then

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$

$$\cos a = \cos b \cos c + \sin\alpha \sin b \sin c$$

$$\cos\alpha = -\cos\beta\cos\gamma + \sin a \sin\beta \sin\gamma$$

J.Rasmussen@dpmms.cam.ac.uk