## SOLUTIONS FOR EXAMPLE SHEET 1

1. Show that any orientable two-manifold has an orientation reversing diffeomorphism.

   $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ has an orientation reversing diffeomorphism given by $(x,y,z) \mapsto (-x,y,z)$. $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ has an orientation reversing diffeomorphism $(x,y) \mapsto (-x,y)$. This diffeomorphism preserves small disks centered on the $y$ axis. Taking connected sum along such disks gives an orientation reversing diffeomorphism for higher genus surfaces.

2. (a) Let $V \subset \mathbb{R}^4$ be the lattice $D_4 \cup (D_4 + v)$, where $v = (1/2, 1/2, 1/2, 1/2)$. Let $Q$ be the quadratic form on $V$ defined by $Q(x,x) = x \cdot x$. Show that $Q \simeq 4\,(1)$.

   The vectors $(1/2, 1/2, 1/2, 1/2)$, $(1/2, -1/2, 1/2, 1/2)$, $(1/2, 1/2, -1/2, 1/2)$, and $(1/2, 1/2, 1/2, -1/2)$ form an orthonormal basis of $V$ with respect to $Q$.

   (b) Show there is a positive definite unimodular form on $\mathbb{Z}^{12}$ which is not isomorphic to $12\,(1)$.

   Consider the lattice $D_{12} \cup (D_{12} + v)$ where $v$ is the vector with all coordinates $1/2$. $D_{12}$ is unimodular (same argument as for $E_8$ in class), but contains no vector of length 1. Elements of $D_{12}$ have length at least 2, and elements of $D_{12} + v$ have length at least $12 \cdot (1/4) = 3$.

3. Show that $(S^2 \times S^2)\#\overline{\mathbb{CP}}^2$ is diffeomorphic to $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. (It might help to understand the isomorphism of intersection forms first.)

   Let $E$ be the obvious sphere of self-intersection $-1$ in $X = (S^2 \times S^2)\#\overline{\mathbb{CP}}^2$, and let $\pi : X \to S^2 \times S^2$ be the blow-down map. Suppose $\pi(E) = p = (p_1, p_2)$. If $\Sigma$ is a curve in $S^2 \times S^2$, we define its *proper transform* $\Sigma'$ to be the closure of $\pi^{-1}(\Sigma - p)$ in the blow-up. We let $S_1, S_2 \subset S^2 \times S^2$ be the spheres $p_1 \times S^2$ and $S_2 = S^2 \times p_1$. Then $S_i' \cdot E = 1$, so $S_i' = S_i - E$, where we have identified $H_2(X) = H_2(S^2 \times S^2) \oplus H_2(\overline{\mathbb{CP}}^2)$. In particular $S_i' \cdot S_i' = -1$. We can now blow down $S_1'$ and $S_2'$ to obtain $\mathbb{CP}^2$.

4. Let $D$ be the unit disk in $\mathbb{C}$, and let $X = D \times 0 \cup 0 \times D \subset D \times D$. Show that $X$ is homologous to a smoothly embedded annulus $A$ with $A \cap \partial(D \times D) = X \cap \partial(D \times D)$.

   $\partial(D^2 \times D^2) = S^3$, and $X \cap \partial(D^2 \times D^2)$ is the Hopf link,which bounds an obvious embedded annulus in $S^3$. Push this into $D^2 \times D^2$.

5. Let $E_n$ be the complex line bundle over $S^2$ with Euler number $n$, and let $D_n$ be its unit disk bundle. Find all the groups and maps in the long exact sequence on homology for the pair $(D_n, \partial D_n)$. Show that the universal cover of $\partial D_n$ is $S^3$. Describe the action of the group of deck transformations.

   $D_n$ is homotopy equivalent to $S^2$, so $H_*(D_n) = \mathbb{Z}$ for $* = 0, 2$ and is 0 otherwise. By Poincare duality and the universal coefficient theorem, $H_*(D_n, \partial D_n) = \mathbb{Z}$ for $* = 4, 2$ and is 0 otherwise. The interesting part of the exact sequence is

$$0 \to H_2(\partial D_n) \to H_2(D_n) \to H_2(D_n, \partial D_n) \to H_1(\partial D_n) \to 0.$$

   The map $H_2(D_n) \to H_2(D_n, \partial D_n)$ is given by multiplication by $n$. To see this, consider intersection numbers in $D_n$. $H_2(D_n, \partial D_n)$ is generated by the class of a fibre, which has intersection number 1 with the 0-section. On the other hand, the 0-section has intersection number $n$ with itself. So it must be homologous to $n$ times the generator.

   Note that $D_{-1}$ is the Hopf bundle, which has total space $S^3$. We can define an $n$ to 1 covering map $\partial D_{-1} \to \partial D_{-n}$ as follows. Let $E_i$ be the total space of the vector bundle with $c_1(E_i) = i$, and consider the map $p_n : E_{-1} \to E_{-1}^{\otimes n} \cong E_{-n}$ which takes a vector $v$ to $v \otimes v \ldots \otimes v$. Observe that $\mathbb{C}^{\otimes n} \cong \mathbb{C}$ via the map which takes $\lambda_1 \otimes \cdots \otimes \lambda_n$ to $\lambda_1 \lambda_2 \ldots \lambda_n$. Thus in local coordinates we have $p_n(\lambda) = \lambda^n$. This map clearly preserves the unit circle in $\mathbb{C}$. It follows that the map $p_n$ takes $\partial D_{-1}$ to $\partial D_{-n}$. It is an $n$ to 1 cover on each fibre of the projection to $S^2$. If we identify the total space of $\partial D_{-1}$ with $S^3$, the projection takes $(z, w)$ to $[z : w]$. The group of deck transformations is generated by the map $(z, w) \mapsto (e^{2\pi i/n} z, e^{2\pi i/n} w)$.

   For $n > 0$, note that $D_n$ is orientation reversing diffeomorphic to $D_{-n}$ via the map that acts as conjugation on each fibre.

6. Compute the genus of a smooth algebraic curve of bidegree $(m, n)$ (*i.e.* representing the homology class $(m, n)$ with respect to the standard basis $\{[\mathbb{CP}^1] \times a, a \times [\mathbb{CP}^1]\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

   Let $X$ be such a curve, and let $i : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the injection. We have

$$\begin{aligned} \chi(X) = c_1(TX) &= i^*(c_1(X)) - c_1(N_{\mathbb{CP}^1 \times \mathbb{CP}^1} X) \\ &= i^*(2PD([\mathbb{CP}^1] \times a) + 2PD(a \times [\mathbb{CP}^1])) - X \cdot X \\ &= 2m + 2n - 2mn \end{aligned}$$

   Thus the genus is $1 - \chi/2 = mn - n - m + 1$.

7. Show that there are two isomorphism classes of three-dimensional real vector bundles over $S^2$. Show that the unit sphere bundle of the nontrivial bundle is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

According to the next problem, such bundles are classified by $\pi_1(SO(3)) \cong \mathbb{Z}/2$, so there is only one nontrivial bundle. To construct such a bundle, consider the rational map $f : \mathbb{CP}^2 \to \mathbb{CP}^1$ defined in homogenous coordinates by $[x : y : z] \mapsto [x : y]$. This map fails to be defined at $p = [0 : 0 : 1]$. If we blow up at $p$, we get a well-defined map, as follows. Pick local coordinates $s = x/z, t = y/z$ near $p$. Local coordinates on the blowup are $([S : T], \lambda)$, where $s = \lambda S$, $t = \lambda T$. For $\lambda \neq 0$

$$f([S : T], \lambda) = [s : t] = [\lambda S : \lambda T] = [S : T].$$

Thus $f$ has a unique continuous extension to the blowup, namely $f([S : T], \lambda) = [S : T]$. If $q = [a : b]$ is a point in $\mathbb{CP}^1$, then $f^{-1}(q)$ consists of the complex line

$$\{[x : y : z] \in \mathbb{CP}^2 \mid bx - ay = 0\}$$

with the point $[0 : 0 : 1]$ removed and replaced by the point $[a : b]$ in the exceptional divisor. In other words, each fibre is a copy of $S^2$. This shows that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is an $S^2$ bundle over $S^2$. Its intersection form is odd, so it must be the nontrivial bundle.

8. Show that $n$ dimensional real vector bundles over $S^m$ are classified by $\pi_{m-1}(O(n))$. Show that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Explain why this makes sense in terms of characteristic classes. Find two four-dimensional vector bundles over $S^4$ whose images in $\mathbb{Z} \oplus \mathbb{Z}$ are linearly independent. Conclude that there are infinitely many vector bundles over $S^4$ with Euler number 1. This is how Milnor constructed the first exotic 7-spheres. (NB: There are only finitely many diffeomorphism classes of manifolds homeomorphic to $S^7$. Different vector bundles can have diffeomorphic unit sphere bundles.)

   Given $\phi : S^{m-1} \to O(n)$, construct a vector bundle with total space

$$E_\phi = \left(D^m_1 \times \mathbb{R}^n \coprod D^m_2 \times \mathbb{R}^n\right) / \sim,$$

   where $(x, y) \in \partial D^m_1 \times \mathbb{R}^n$ is identified with $(x, \phi(x) \cdot y) \in \partial D^m_2 \times \mathbb{R}^n$. Suppose $E_\phi$ is isomorphic to $E_\psi$. After composition with a further isomorphism, we can assume that the isomorphism is identity on the northern hemisphere. In this case, it is not difficult to see that the isomorphism extends to the southern hemisphere if and only if the map $\phi \circ \psi^{-1}$ extends to the southern hemisphere. This occurs if and only $[\phi] - [\psi] = 0$in $\pi_{m-1}(O(n))$. (Proof: after a homotopy, we can assume that $\phi$ and $\psi$ are equal to $I$ outside of small disjoint disks in $S^{m-1}$. In this case it is easy to see that multiplication in $O(n)$ and addition in the homotopy group coincide.)

   To compute $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$, we can either use the isomorphism $SO(4) \cong (SU(2) \times SU(2))/\{\pm(I, I)\}$ or the long exact sequence of the fibration $SO(3) \to SO(4) \to S^3$ and the isomorphism $SO(3) \cong SU(2)/\{\pm I\}$. The fact that the set of $SO(4)$ bundles over

$S^4$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ relates to the fact that there are two different characteristic classes for such a bundle $E$. Namely, $p_1(E)$ and $e(E)$, both of which are potentially nontrivial elements of $H^4(S^4) = \mathbb{Z}$.

9. Show that a smooth degree 2 hypersurface in $\mathbb{CP}^3$ is diffeomorphic to $S^2 \times S^2$, and that a smooth cubic hypersurface is diffeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$.

   Let $Q$ be the quadric hypersurface determined by the equation $xy = zw$ in $\mathbb{CP}^3$. There is a map $f : Q \to \mathbb{CP}^1$ defined by $[x : y : z : w] \mapsto [x : z]$.

   $$f^{-1}([a : b]) = \{[x : y : z : w] \mid az - bx = 0, ay - bw = 0\}$$

   is a complex line in $\mathbb{CP}^2$, so it is homeomorphic to $S^2$. Thus $Q$ is an $S^2$ bundle over $S^2$. To see that it is the trivial $S^2$ bundle, not that $c_1(Q)$ is even, so the intersection form on $Q$ is even.

   For the cubic hypersurface, we proceed as follows. Choose 6 points in $p_1, \ldots, p_6$ in $\mathbb{CP}^2$ with the property that there is a 4-dimensional vector space $V$ of cubic polynomials which vanish at all 6. To see that 4 is the right number to expect here, notice that the space of all homogenous cubic polynomials is 10 dimensional, and that vanishing at each point imposes a single linear condition. (Note, however, that this does not prove that such a collection of 6 points exists. The lazy way to do this is just to pick 6 random points, and check, using Mathematica or the like, that the resulting system of 6 linear equations in 10 variables has a 4 dimensional space of solutions. For a better argument, look in a book on algebraic surfaces.) Pick a basis $P_1, P_2, P_3, P_4$ for $V$, and consider the rational map $f : \mathbb{CP}^2 \to \mathbb{CP}^3$ defined by

   $$\mathbf{x} = [x_1 : x_2 : x_3] \mapsto [P_1(\mathbf{x}) : P_2(\mathbf{x}) : P_3(\mathbf{x}) : P_4(\mathbf{x})].$$

   After blowing up at $p_1, \ldots, p_6$, we get a well-defined map $f : \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2 \to \mathbb{CP}^3$.

   Any huypersurface in $\mathbb{CP}^n$ is the zero locus of a single equation, so we need only determine the degree of $S = \text{im} f$. To do this, consider the intersection of $S$ with the line $L = [z_1 : z_2 : 0 : 0]$ in $\mathbb{CP}^3$; that is, the set of points in $S$ with $P_3(\mathbf{x}) = P_4(\mathbf{x})$. $P_3$ and $P_4$ each define a cubic hypersurface in $\mathbb{CP}^2$, so they intersect at nine points. Six of these are the blow-up points $p_1, \ldots, p_6$, and do not appear in $L \cap S$. The other three points do, so $S$ has degree 3 in $\mathbb{CP}^3$.

10. What is the intersection form of a smooth hypersurface of degree 5 in $\mathbb{CP}^3$? Same question for degree 6.

    Suppose $S$ be a smooth hypersurface of degree $n$ in $\mathbb{CP}^3$, and let $i : S \to \mathbb{CP}^3$ be the inclusion. Let $H = PD_{\mathbb{CP}^3}([\mathbb{CP}^2])$ be a generator of $H^2(\mathbb{CP}^3)$, and let $h = i^*(H)$. Then we have

$$\begin{aligned}
i^*(T\mathbb{CP}^3) &= TS \oplus N_{\mathbb{CP}^3}S \\
i^*(c(T\mathbb{CP}^3)) &= c(S) \cup c(N_{\mathbb{CP}^3}S) \\
1 + 4h + 6nPD_S(1) &= (1 + c_1(S) + c_2(S)) \cup (1 + nh)
\end{aligned}$$

Solving, we find that $c_1(S) = (4-n)h$ and that

$$c_2(S) = 6nPD_S(1) + (n-4)nh^2 = n(n^2 - 4n + 6)PD_S(1)$$

The intersection form is even if and only if $w_2(S) \equiv 0(2)$, which occurs if and only if $n$ is even. $S$ is simply connected by the Lefshetz hyperplane theorem, so $b_2(S) = \chi(S) - 2 = c_2(S) - 2$. When $n = 5$, we have $b_2 = 55 - 2 = 53$. To compute $\sigma(S)$ we use the signature formula

$$\sigma(S) = \frac{p_1(S)}{3} = \frac{c_1^2(S) - 2c_2(S)}{3}$$

For $n = 5$, this works out to $(5 - 110)/3 = -35$, so the intersection form is $9(1) \oplus 44(-1)$.

When $n = 6$, we have $b_2 = 6(42 - 24) - 2 = 106$ and $\sigma = (24 - 216)/3 = -64$, so the intersection form is $21H \oplus 8E_8$.

11. (a) Let $Q$ be a unimodular quadratic form on a real vector space $V$. Suppose that $V$ has a subspace $L$ whose dimension is half that of $V$ and that $Q(x, y) = 0$ for all $x, y \in L$. Show that $\sigma(Q) = 0$.

    Let $V$ have dimension $2n$, and let $V_+$ and $V_-$ be maximal positive and negative definite subspaces, so that $\dim V_+ + \dim V_- = 2n$. We claim that $\dim V_+ = \dim V_- = n$. Indeed, if this were not the case, either $V_+ \cap L$ or $V_- \cap L$ would contain a nonzero element, which is impossible.

    (b) Suppose that $M$ is a $4k$-manifold which bounds a $(4k+1)$-manifold $W$. Use the long exact sequence of the pair $(W, M)$ to show that $\sigma(M) = 0$.

    Let $i : M \to W$ be the inclusion. Then

$$\begin{aligned}
\langle i^*(x) \cup i^*(y), [M] \rangle &= \langle x \cup y, i_*([M]) \rangle \\
&= \langle x \cup y, 0 \rangle \\
&= 0
\end{aligned}$$

    so it suffices to show that $i^*(H^{2k}(W))$ has half the dimension of $H^{2k}(M)$. Consider the commutative diagram

$$\begin{tikzcd}
H^{2k}(W,M) \arrow[r] \arrow[d] & H^{2k}(W) \arrow[r, "f_1"] \arrow[d] & H^{2k}(M) \arrow[r, "f_2"] \arrow[d] & H^{2k+1}(W,M) \arrow[r] \arrow[d] & {} \\
H_{2k+1}(W) \arrow[r] & H_{2k+1}(W,M) \arrow[r, "g_1"] & H_{2k}(M) \arrow[r, "g_2"] & H_{2k}(W) \arrow[r] & {}
\end{tikzcd}$$

where the vertical maps are Poincare duality. Then clearly $f_i$ and $g_i$ have the same rank. On the other hand, if we use rational coefficients, $f_1$ is the dual map to $g_2$ and $g_2$ is the dual map to $g_1$. Thus all four maps have the same rank.

12. Let $K \subset S^3$ be a smoothly embedded knot with tubular neighborhood $U$, and let $Y = S^3 - U$ be its complement. Compute all groups and maps in the long exact sequence for the homology of the pair $(Y, \partial Y)$. Show that up to isotopy there is a unique embedded curve $\ell$ on $\partial Y$ which bounds in $Y$. This curve is called the *longitude* of $K$.

First consider the Mayer-Vietoris sequence for the decomposition $S^3 = Y \cup_{T^2} S^1 \times D^2$. From

$$0 \to H_3(S^3) \to H_2(T^2) \to H_2(S^1 \times D^2) \oplus H_2(Y) \to H_2(S^3) = 0$$

we see that $H_2(Y) = 0$, and from

$$0 \to H_1(T^2) \to H_1(S^1 \times D^2) \oplus H_1(Y) \to H_1(S^3) = 0$$

we see that $H_1(Y) = \mathbb{Z}$. By Poincare duality and the universal coefficient theorem, we have $H_1(Y, \partial) = H^2(Y) = 0$ and $H_2(Y, \partial) = H^1(Y) = \mathbb{Z}$. Thus the long exact sequence of the pair

$$0 \to H_3(Y, T^2) \to H_2(T^2) \to H_2(Y) \to H_2(Y, T^2) \to H_1(T^2) \to H_1(Y) \to H_1(Y, T^2) = 0.$$

becomes

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

In particular, the kernel of the map $H_1(T^2) \to H_1(Y)$ is isomorphic to $\mathbb{Z}$. Up to sign, there is a unique primitive element of the kernel. This gives the unique embedded *connected* curve which is nontrivial in $H_*(T^2)$ and which bounds on $Y$.

13. Let $X = S^1 \times D^2$, and let $a = S^1 \times 1$ and $b = 1 \times \partial D^2$ be curves on $\partial X = T^2$. Show that a diffeomorphism $f : \partial X \to \partial X$ extends to $X$ if and only if $f_*([b]) = [b]$. (If you like, you may assume $f$ is given by a linear map on $\mathbb{R}^2$.)

This is false as stated; the condition should be $f_*([b]) = \pm[b]$. To see that this condition is necessary, suppose $f$ extends to $X$ and let $i : \partial X \to X$ be the inclusion. Then

$$i_*(f_*([b])) = f_*(i_*([b])) = f_*(0) = 0$$

so $f_*([b])$ is a primitive homology class in $\partial X$ which bounds in $X$. It follows from the previous exercise that $\pm[b]$ are the only elements of $H_2(\partial X)$ with this property.

Conversely, suppose we are given a diffeomorphism $f : \partial X \to \partial X$ which is induced by a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, and for which $f_*([b]) = \pm[b]$. Since $f$ is a diffeomorphism, it preserves the cup product. Thus it preserves intersection numbers up to sign. In particular,$f_*([a]) \cdot f_*([b]) = \pm 1$. This implies that $f_*([a] = \pm[a] + n[b]$ for some $n \in \mathbb{Z}$. It follows that

$$A = \begin{bmatrix} \pm 1 & 0 \\ n & \pm 1 \end{bmatrix}$$

Suppose that the $\pm 1$'s in this expression are both $+1$. (The other cases can be easily dealt with by composing with the conjugation map in either the $S^1$ or the $D^2$ factor, or both.) Then $f(z, w) = (z, z^n w)$. The same formula defines the desired extension of $f$ to all of $S^1 \times D^2$.

14. In the notation of problems 12 and 13, let $M = Y \cup_f X$, where $f : \partial X \to \partial Y$ is an orientation reversing diffeomorphism with $f_*([b]) = p[m] + q[\ell]$. Show that if $f'$ is any other such $f$, the resulting manifold $M'$ is diffeomorphic to $M$. ($M$ is called $p/q$ surgery on $K$.) Compute $H_*(M)$.

    By the previous problem, $g = f' \circ f^{-1}$ has $g_*([b]) = [b]$, so $g$ extends to a diffeomorphism $G : X \to X$. We can use $G$ to define a diffeomorphism $h : M \to M'$. $h$ is the identity on $Y$ and acts by $G$ on $X$. To compute $H_*(M)$, note that by the Seifert Van-Kampen theorem, adding $X$ to $Y$ has the effect of killing the loop $p[m] + q[\ell]$ on the boundary torus. $\ell$ bounds in $Y$, so the effect on $H_1$ is to kill $p[m]$, where $[m]$ generates $H_1(Y) = \mathbb{Z}$. Thus $H_1(M) = \mathbb{Z}/p$. By Poincare duality and the universal coefficient theorem, $H_2(M) = 0$.

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