Knots and 4-Manifolds
Sample Exam

Attempt BOTH questions 1 and 2 and ANY THREE of questions 3-7. (Five questions in total.)

1. Let K be the knot shown in Figure 1. Find a presentation for $\pi_1(S^3) - K$. Using Fox calculus, compute $\Delta_K(t)$.

2. Compute the Alexander polynomial, Seifert genus, signature, and slice genus of the knot shown in Figure 2. (You may use any result from lectures or the example sheets.)

3. Let $f : B^{n-k} \times S^{k-1} \to \partial X$ be the attaching map which specifies the attachment of a k-handle to X^n. What is the *attaching sphere* of this map? Prove or give a counterexample to the following statement: any embedded submanifold of ∂X which is diffeomorphic to S^{k-1} is the attaching sphere of some attaching map.

   Explain what is meant by the *framing* of the attaching map. What group are framings classified by? When $X = B^4$ and $k = 2$, any one-dimensional submanifold of ∂X has a canonical framing. What is it? Explain how framings are related to the self-intersection of embedded surfaces in this case.

4. Let K be the (p, q) torus knot. Use Seifert Van-Kampen to find a presentation of $\pi_1(S^3 - K)$ with two generators and one relation. Compute the Alexander polynomial of K and show that the Seifert genus of K is $(p - 1)(q - 1)/2$.

5. Explain how to compute the Alexander polynomial and signature of a knot K if you are given a Seifert matrix of K. (No proofs are needed, just statements.) Show that $\sigma(K) \equiv 0 \bmod(4)$ if $\Delta_K(-1) < 0$, and that $\sigma(K) \equiv 2 \bmod(4)$ if $\Delta_K(-1) > 0$. Show moreover that if K_+ is related to K_- by changing a single crossing from positive to negative, then $\sigma(K_+) \leq \sigma(K_-) \leq \sigma(K_+) + 2$.

6. Define the smooth four-ball genus of a knot K. Give an example (with justification) of a nontrivial knot with smooth four-ball genus 0. Let $S \subset B^4$ be an embedded surface with boundary K. Given that the branched double cover $D(S)$ has $b_1 = 0$, show that $2\sigma(K) \leq g_*(K)$.

7. Let D be a planar diagram representing a knot K. What is the *cube of resolutions* of D? Define the Khovanov homology of K. (You need not check that it is well defined). Suppose that D is an alternating diagram with no *nugatory* crossings. (That is, no crossings of the form shown in Figure 3.) Show that the Khovanov homology of such a diagram is nontrivial in both the highest and lowest homological gradings in the cube of resolutions.

(Hint: show that the number of circles at the bottom vertex of the cube is greater than the number of circles at any adjacent vertex.)Conclude that no planar diagram representing K has fewer crossings than D.