Proof of Seifert-Van Kampen

Setup: $X = U_1 \cup U_2$, $U_i \subset X$ open, $U_1 \cap U_2$ path connected.
Basepoint $x \in U_1 \cap U_2$.
Let $j_i : U_1 \cap U_2 \to U_i$ be the inclusion.
Let $G_i = \pi_1(U_i, x)$.

Theorem: $\pi_1(X, x) \cong G_1 * G_2/N$ where N is the smallest normal subgroup of $G_1 * G_2$ containing all elements of the form $j_{1*}(a)j_{2*}(a)^{-1}$ for all $a \in \pi_1(U_1 \cap U_2, x)$.

Proof:
We have already established (easy SVK) that there is a surjection $\phi : G_1 * G_2 \to \pi_1(X, x)$, and that $N \subset \ker\phi$. It remains to show that there are no other elements of the kernel.

Suppose $\gamma = \gamma_1\gamma_2 \dots \gamma_n$ is represented by a composition of loops γ_i based at x, where each γ_i is contained in U_1 or U_2. If γ can be written as a composition of paths $f_1 \dots f_n$, where each f_i is contained in U_1 or U_2, then we can represent γ by the composition $(f_1g_1^{-1})(g_1f_2g_2^{-1})\dots(g_{n-1}f_n) \in G_1 * G_2$, where g_i is a path from x to $f_{i-1}(1) = f_i(0)$.

Claim 1: $\gamma_1\gamma_2 \dots \gamma_n = (f_1g_1^{-1})(g_1f_2g_2^{-1})\dots(g_{n-1}f_n)$ in $G_1 * G_2/N$.
Proof: It suffices to consider the case where we insert a single $g_i^{-1}g_i$ into some word γ_j, which we assume is contained in U_1. In this case, we have replaced γ_j with $(\gamma_{j,1}g_i^{-1})(g_i\gamma_{j,2})$. If both words in this expression are considered as elements of G_1, then the desired relation already holds in $G_1 \cap G_2$. If one or both are considered as elements of G_2, then they must be contained in $U_1 \cap U_2$, and the desired relation is a consequence of the relations in N.

Now suppose that γ and γ' are loops in X which are related by a homotopy $H : [0, 1] \times [0, 1] \to X$. In light of claim 1, γ and γ represent well-defined elements of $G_1 * G_2/N$.

Claim 2: Suppose $[0, 1]$ can be divided into intervals $[a_0, a_1], [a_1, a_2], \dots [a_{n-1}, a_n]$, where $a_0 = 0, a_n = 1$, and that $H([a_i, a_{i+1}], [0, 1])$ is contained in U_1 or U_2 for each i. Then $\gamma = \gamma'$ in $G_1 * G_2/N$.
Proof: Let f_i be the restriction of γ to the interval $[a_i - 1, a_i]$, and similarly for f_i' and γ'. Then we can write

$$\gamma = ((f_1g_1^{-1})(g_1f_2g_2^{-1})\dots(g_{n-1}f_n) \in G_1 * G_2/N$$

, where g_i is a path from x to $f_{i-1}(1) = f_i(0)$. Similarly,

$$\gamma' = ((f_1'h_1^{-1}g_1^{-1})(g_1h_1f_2'h_2^{-1}g_2^{-1})\dots(h_{n-1}g_{n-1}f_n') \in G_1 * G_2/N$$

where h_i is the path obtained by restriction H to the interval $a_i \times [0, 1]$. Since the homotopy $H)([a_i, a_{i+1}] \times [0, 1]$ is entirely contained in either U_1 or U_2, the relation

$$g_{i-1}f_ig_i^{-1} = g_{i-1}h_{i-1}f_i'h_i^{-1}g_i^{-1}$$

holds in G_1 or G_2. Combining these relations for all i gives the claim.

We can now complete the proof of the theorem. Suppose that γ and γ' are loops in X which are related by a homotopy $H : [0,1] \times [0,1] \to X$. By the uniform continuity lemma, we can choose an integer N so that each square of side $1/N$ in $[0,1] \times [0,1]$ maps to U_1 or U_2 under H. Divide the square into smaller squares of side $1/N$, and let γ_i be the restriction of H to $[0,1] \times i/N$. Then claim 2 ensures that γ_i and γ_{i+1} represent the same element of $G_1 * G_2/N$. Repeating, we see that $\gamma = \gamma'$ in $G_1 * G_2/N$.

Corollary: Suppose we have presentations

$$\pi_1(U_1, x) = \langle a_1, \ldots a_n \mid w_1, \ldots w_m \rangle$$
$$\pi_1(U_2, x) = \langle b_1, \ldots b_{n'} \mid u_1, \ldots u_{m'} \rangle$$

and that $\pi_1(U_1 \cap U_2, x)$ is generated by elements $c_1, \ldots c_k$. Then

$$\pi_1(X) \cong \langle a_1, \ldots a_n, b_1, \ldots b_{n'} \mid w_1, \ldots w_m,\ u_1, \ldots u_{m'}, j_{1*}(c_1) j_{2*}(c_1)^{-1} \ldots j_{1*}(c_k) j_{2*}(c_k)^{-1} \rangle$$

Proof: This follows easily from the fact that

$$G_1 * G_2 \cong \langle a_1, \ldots a_n, b_1, \ldots b_{n'} \mid w_1, \ldots w_m,\ u_1, \ldots u_{m'} \rangle.$$

Example 1: $T^2 = U_1 \cup U_2$, where U_1 deformation retracts to $S^1 \vee S^1$, $U_2 \simeq (B^2)^\circ$, and $U_1 \cap U_2 \simeq S^1 \times (0,1)$. $pi_1(U_1) = \langle a, b \mid \rangle$, $\pi_1(U_2)$ is trivial, and $\pi_1(U_1 \cap U_2)$ is generated by a simple loop γ which deformation retracts to the loop $aba^{-1}b{-}1 = [a,b]$ in $S^1 \vee S^1$. It follows that

$$\pi_1(T^2, x) \cong \langle a, b \mid ab = ba \rangle = \mathbb{Z}^2$$

Example 2: $\mathbb{RP}^2 = U_1 \cup U_2$, where U_1 is the Mobius band, $U_2 = (B^2)^\circ$, and $U_1 \cap U_2 = S^1 \times (0,1)$. $\pi_1(U_1) = \langle a \mid \rangle$, and $\pi_1(U_1 \cap U_2)$ is generated by a loop γ which represents a^2 in $\pi_1(U_1)$. It follows that

$$\pi_1(T^2, x) \cong \langle a \mid a^2 = 1 \rangle = \mathbb{Z}/2$$

Both of these examples are special cases of the construction of attaching a disk: if $f : S^1 \to X$, then $X \cup_f B^2 = X \coprod B^2/\sim$, where $x \sim f(x)$.

Example 3: Σ_2 is a surface of genus 2 obtained by identifying two copies of $T^2 - B^2$ along their common boundary circle. Taking U_1 and U_2 to be slightly enlarged copies of the punctured tori, we see that $U_1 \cap U_2 \simeq S^1 \times (-1,1)$. We have $\pi_1(U_i) = \langle a_i, b_i \mid \rangle$. If γ is a generator of $\pi_1(U_1 \cap U_2)$, we see that $j_{1*}(\gamma) = [a_1, b_1]$ and $j_{2*}(\gamma) = [a_2, b_2]^{-1}$. Thus

$$\pi_1(\Sigma_2, x) \cong \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle$$

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