Products

Set Theory Background: If X and Y are sets,

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

If $A \subset X, B \subset Y$, $A \times B \subset X \times Y$. There are *projections* $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$.

Topology: If X and Y are topological spaces, the *product topology* on $X \times Y$ is defined as follows: $U \subset X \times Y$ is open if for every $x \times y \in U$ there exist open subsets $V_x \subset X$, $V_y \subset Y$, with $x \in V_x$, $y \in V_y$, and $V_x \times V_y \subset U$.

If U is open in $X \times Y$, then

$$U = \bigcup_{x \times y \in U} V_x \times V_y$$

where V_x and V_y are as in the definition. Thus U is open if and only if it is union of sets of the form $V_x \times V_y$, where V_x is open in X and V_y is open in Y.

Set Theory Fact:

$$\bigcap_{i=1}^{n} (A_i \times B_i) = \left(\bigcap_{i=1}^{n} A_i \right) \times \left(\bigcap_{i=1}^{n} B_i \right)$$

Proof it's a topology:

(1) $\emptyset$ is open in X and Y, so $\emptyset = \emptyset \times \emptyset$ is open in $X \times Y$, X is open in X and Y is open in Y, so $X \times Y$ is open in $X \times Y$.

(2) Suppose $x \times y \in W = \bigcup_{\alpha \in A} U_\alpha$, where each U_α is open in $X \times Y$. Then $x \times y \in U_\alpha$ for some α, so there are open subsets $V_x \subset X$, $V_y \subset Y$, $x \in V_x$, $y \in V_y$ with $V_x \times V_y \subset U_\alpha \subset W$. It follows that W is open in $X \times Y$.

(3) Suppose $x \times y \in W = \bigcap_{i=1}^{n} U_i$, where each U_i is open in $X \times Y$. Then $x \times y \in U_i$ so there are open subsets $V_{x,i} \subset X$, $V_{y,i} \subset Y$, $x \in V_{x,i}$, $y \in V_{y,i}$ with $V_{x,i} \times V_{y_i} \subset U_i$. So

$$x \times y \in \bigcap_{i=1}^{n} (V_{x,i} \times V_{y,i}) \subset \bigcap_{i=1}^{n} U_i$$

and

$$\bigcap_{i=1}^{n} (V_{x,i} \times V_{y,i}) = \left(\bigcap_{i=1}^{n} V_{x,i} \right) \times \left(\bigcap_{i=1}^{n} V_{y,i} \right)$$

is of the form $W_x \times W_y$, where W_x is open in X and W_y is open in Y. Thus W is open in $X \times Y$.

Universal Property: $f : Z \to X \times Y$ is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Set Theory Fact: $f^{-1}(A \times B) = ((\pi_1 \circ f)^{-1}(A)) \cap ((\pi_2 \circ f)^{-1}(B))$

Proof. π_1 is continuous, since if $V \subset X$ is open, $\pi_1^{-1}(V) = V \times Y$ is open in $X \times Y$. Similarly for π_2. Thus if f is continuous, $\pi_i \circ f$ is continuous.

Conversely, if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, and V_x is open in X and V_y is open in Y, then

$$f^{-1}(V_x \times V_y) = ((\pi_1 \circ f)^{-1}(V_x)) \cap ((\pi_2 \circ f)^{-1}(V_y))$$

is the intersection of two open sets, hence open in Z. If U is any open set in $X \times Y$, then

$$U = \bigcup_{x \times y \in U} V_x \times V_y \quad \text{so} \quad f^{-1}(U) = \bigcup_{x \times y \in U} f^{-1}(V_x \times V_y)$$

is a union of open sets in Z, hence open. It follows that f is continuous.

Quotients

Set Theory Background: If $\sim$ is an equivalence relation on X, the quotient $X/\sim$ is the set of equivalence classes for $\sim$. There is a natural projection $\pi : X \to X/\sim$ which sends an element of X to its equivalence class.

Topology: If X is a topological space, the *quotient topology* on $X/\sim$ is defined as follows: $U \subset X/\sim$ is open if and only if $\pi^{-1}(U)$ is open.

Set Theory Fact:

$$f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(U_\alpha)$$

Proof it's a topology:

(1) $\pi^{-1}(\emptyset) = \emptyset$ is open in X, so $\emptyset$ is open in $X/\sim$. $\pi^{-1}(X/\sim) = X$ is open in X, so $X/\sim$ is open in $X/\sim$.

(2) If U_α is open in $X/\sim$, then

$$f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$$

is a union of open sets in X, so it is open in X. Thus $\cup_{\alpha \in A} U_\alpha$ is open in $X/\sim$.

(3) If U_i is open in $X/\sim$, then

$$f^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} f^{-1}(U_i)$$

is a finite intersection of open sets in X, so it is open in X. Thus $\bigcap_{i=1}^{n} U_i$ is open in $X/\sim$.

Universal Property: $f : X/\sim \to Y$ is continuous if and only if $f \circ \pi$ is continuous.

Proof. If U is open in $X/\sim$, then by definition $\pi^{-1}(U)$ is open in X. So π is continuous. Hence if f is continuous, $f \circ \pi$ is too.

Conversely, if $f \circ \pi$ is continuous and U is open in Y, then $(f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ is open in X. It follows that $f^{-1}(U)$ is open in $X/\sim$, so f is continuous.

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