Morse Theory **Lent 2016**

Example Sheet 4

1. A map $p : E \to B$ is called a *Serre fibration* if for any continuous maps $f : D^n \times [0,1] \to B$ and $g : D^n \to E$ with $p \circ g = f|_{D^n \times 0}$, there is a continuous map $F : D^n \times [0,1] \to E$ with $p \circ F = f$ and $F|_{D^n \times 0} = g$. Show that the map $\mathcal{P}(X, *) \to X$ which sends γ to $\gamma(1)$ is a Serre fibration.

2. If $p : E \to B$ is a Serre fibration and $* \in E$, let $*_B = p(*)$ and $F = p^{-1}(*_B)$. Let $X = S^{n-1} \times [0,1]$, $A = S^{n-1} \times \{0,1\} \cup * \times [0,1] \subset X$, and let $\pi : X \to X/A \simeq S^n$ be the projection map. If $\phi : (S^n, *) \to (B, *_B)$, show that we can find a map $\Phi : X \to E$ such that $p \circ \Phi = \phi \circ \pi$ and $\Phi(S^{n-1} \times 0) = *$. If we define $\partial\phi = \Phi|_{S^{n-1} \times 1} : S^{n-1} \to F$, check that the map $[\phi] \mapsto [\partial\phi]$ descends to a well-defined homomorphism $\partial : \pi_n(B, *_B) \to \pi_{n-1}(F, *)$.

3. Show that the sequence $\to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to$ is exact.

4. Let V be a $2n$-dimensional vector space equipped with a symplectic form ω. Show that there is a basis $\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ of V such that $\omega = \sum_{i=1}^n x_i^* \wedge y_i^*$, where $\langle x_1^*, \dots, x_n^*, y_1^*, \dots y_n^* \rangle$ is the dual basis. Deduce that V admits a compatible almost complex structure.

5. Let M^{2n} be a manifold, and let g and ω be a Riemannian metric and symplectic form on M, respectively. Show that there are maps $A_p : T_pM \to T_pM$ such that if $\mathbf{v}, \mathbf{w} \in T_pM$, $\omega(\mathbf{v}, \mathbf{w}) = \langle A_p\mathbf{v}, \mathbf{w} \rangle$. If $\langle \mathbf{v}_1, \dots, \mathbf{v}_{2n} \rangle$ is an orthonormal basis of T_pM and A is the matrix of A_p with respect to this basis, then A is skew symmetric. Show that there is a symmetric matrix Q with $Q^2 = -A^2$, and that $J = Q^{-1}A$ satisfies $J^2 = -I$. Deduce that there is a bundle map $Q : TM \to TM$ such that $J = Q^{-1}A$ defines an almost complex structure on TM compatible with ω.

6. If M is symplectic and $L \subset M$ is Lagrangian, show that $\nu_L \simeq T^*L$. Deduce that if $\psi_t : M \to M$ is the flow of a vector field on M and L is transverse to $\psi_t(L)$, then $L \cap \psi_t(L)$ contains at least $|\chi(L)|$ points.

7. If S is a closed surface and ω is an area form on S, then ω is a symplectic form, and any simple closed curve $C \subset S$ is a Lagrangian. Show directly that if $\psi_t : (S, \omega) \to (S, \omega)$ is a Hamiltonian isotopy, then $\psi_t(C) \cap C$ contains at least 2 points. Give an example of an isotopy $\psi_t : S \to S$ such that $\psi_t^*(\omega) = \omega$ for all t, but $\psi_t(C) \cap C = \emptyset$ for some simple closed curve $C \subset S$ and some $t > 0$. Check directly that your isotopy is not Hamiltonian.

8. Suppose $\phi \in \pi_2(x, y)$ and $\psi \in \pi_2(y, z)$. Show that $\mu(\phi + \psi) = \mu(\phi) + \mu(\psi)$.

9. Suppose $\pi_2(x, y)$ contains a unique element ϕ, and that $\Gamma \in \mathcal{M}(\phi)$. Show that

$$A(x) - A(y) = \frac{1}{2} \int_S \left| \frac{\partial \Gamma}{\partial u} \right|^2 + \left| \frac{\partial \Gamma}{\partial t} \right|^2 .$$

10. Suppose that $M = \mathbb{C}$, and that L_1 and L_2 are as shown in Figure 1. If the image of $\phi \in \pi_2(x, y)$ is the shaded region, check carefully that $\mathcal{M}(\phi) \simeq \mathbb{R}$. (You may use the fact that if $U \subset \mathbb{C}$ is simply connected and has piecewise smooth boundary, then the holomorphic map int $D^2 \to U$ whose existence is guaranteed by the Riemann mapping theorem extend to a continuous map $D^2 \to \overline{U}$ which send $\partial D^2 \to \partial U$.)

11. Now suppose that L_1 and L_2 are as shown in Figure 2. Show directly $\mu(x, y) = 2$. Can you describe the elements of $\widehat{\mathcal{M}}(\phi)$ geometrically? What are the ends of $\widehat{\mathcal{M}}(\phi)$? How is your answer related to the fact that $d^2 = 0$ in $CL(L_1, L_2)$?

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