Morse Theory **Lent 2016**

Example Sheet 3

1. Suppose $\gamma : [0,1] \to M$ is a smooth path, and that $X \in \Gamma(\gamma^* TM)$ is a piecewise smooth vector field on γ. Let $\{t_1, \ldots, t_k\}$ be the set of points at which X fails to be smooth. Show that there is a continuous variation Γ of γ which is smooth on $[t_i, t_{i+1}] \times (-\epsilon, \epsilon)$ for all i, and which realizes X.

2. If M is connected, show that $\Omega^{ps}(p,q) \sim \Omega^{ps}(p,q')$ for any $q, q' \in M$.

3. If $\gamma : [0,1] \to M$ is a geodesic, show that $t\gamma'(t) \in J_\gamma$.

4. Suppose $p \in S^n$ with the round metric. Show that the only points in S^n which are conjugate to p are $-p$ and p itself. If γ is a geodesic joining p to $-p$ or p, show that $\dim J_\gamma = n - 1$.

5. Suppose γ is a geodesic, and that V_1 is the space of broken Jacobi fields on γ, as defined in class. Show that if $X \in V_1$ and $H_\gamma(X,Y) = 0$ for all $Y \in V_1$, then $X \in J_\gamma$.

6. The *injectivity radius* of M at a point $p \in M$ is the smallest value of r for which $\exp_p : B_r(T_pM) \to M$ fails to be injective. If M is simply connected, show that its injectivity radius at p is equal to the distance between p and the closest point conjugate to p.

7. M has *nonpositive sectional curvature* if for all $p, \in M$ and all $\mathbf{v}, \mathbf{w} \in T_pM$, we have $\langle R(\mathbf{v},\mathbf{w})\mathbf{v}, \mathbf{w}\rangle \leq 0$. Suppose that M has nonpositive sectional curvature, that $\gamma : [0,1] \to M$ is a geodesic, and that $X \in J_\gamma$. Show that $\frac{d}{dt}\langle \nabla_{\partial/\partial t} X, X\rangle \geq 0$. Deduce that p has no conjugate points. If M is simply connected, show that M is diffeomorphic to $\mathbb{R}^n$.

8. Show that $\Omega(S^{n+1})$ is homotopy equivalent to a cell complex obtained by starting with S^n and adding cells of dimension $\geq 2n$. Deduce that $\pi_{i+1}(S^{n+1}) \simeq \pi_i(S^n)$ for $i \leq 2n-2$, and hence that $\lim_{n\to\infty} \pi_{i+n}(S^n)$ exists.

9. Suppose X and Y are left-invariant vector fields on $GL_m(\mathbb{R})$. Show that $[X,Y]|_I = [X_I, Y_I]$, where on the left $[\cdot,\cdot]$ denotes the Lie bracket of vector fields, and on the right it denotes the commutator of matrices.

10. Suppose $G = SU(3)$, and that $g = \exp x \in G$. By considering the Cartan subalgebra of G, describe the set of geodesics from I to g. There are several possible cases to consider, depending on the position of x relative to the walls of the Weyl chamber. In each case, determine the index of the relevant geodesics and thus compute $H_*(\Omega G)$. (You should get the same answer in each case, of course, but the computation will look somewhat different from case to case.)

11. As in the previous question, but for $G = SU(n)$ and now assuming that g is in generic position with respect to the walls of the Weyl chamber.

12. For those who know a bit more about Lie groups and Lie algebras, try doing the analog of question 10 for $G = Spin(4)$ and $G = G_2$.

13. (Optional, and definitely non-examinable, but still interesting.) Let G be a Lie group, and let BG be the classifying space for G bundles. (So if $G = U(n)$, BG is the Grassmannian of n-planes in C^∞.) If T is a maximal torus for G, show that there is a map from $H^*(BG) \to H^*(BT)$, and that the image of this map is invariant under the Weyl group. (In fact, $H^*(BG; \mathbb{Q})$ is isomorphic to the ring of invariants.) What is the ring of invariants when $G = SU(n)$? When $G = SO(4)$? when $G = G_2$?. How are these related to your answers to the previous three questions?

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