Morse Theory **Lent 2016**

Example Sheet 2

1. Sketch a sphere S^2 embedded in $\mathbb{R}^3$ such that the z coordinate is a Morse function with two index 2 critical points, one index 1 critical point, and one index 0 critical point. Sketch the ascending and descending disk of each critical point. What is the Morse complex?

2. Suppose $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$ are Morse. Show that $g = f_1 \circ \pi_1 + f_2 \circ \pi_2$ is a Morse function on $M_1 \times M_2$, where π_1 and π_2 are the natural projections. Assuming that f_1 and f_2 satisfy the Palais-Smale condition (all attaching spheres and belt spheres are transverse), show that $C_*^g(M_1 \times M_2; \mathbb{Z}/2) \simeq C_*^{f_1}(M_1; \mathbb{Z}/2) \otimes C_*^{f_2}(M_2; \mathbb{Z}/2)$. (This also works over $\mathbb{Z}$, but I won't force you to think about the signs.)

3. Suppose $\iota : S^{k-1} \times D^{n-k} \to M^{n-1}$ is an embedding, and let $A = \iota(S^{k-1} \times 0)$. Show that $\iota \simeq e \circ T \circ j_\delta$, where

$$j_\delta : S^{k-1} \times D^{n-k} \to S^{k-1} \times B_\delta \subset S^{k-1} \times \mathbb{R}^{n-k}$$

is the inclusion, $T : S^{k-1} \times \mathbb{R}^{n-k} \to \nu_A$ is a bundle isomorphism, and $e : \nu_A \to M$ is the exponential map with respect to a fixed Riemannian metric on M. (Hint: first reduce to the case that $M = S^{k-1} \times D^{n-k}$, and $\iota_{S^{k-1} \times 0}$ is the identity map. Show that $\iota \simeq \iota_1$, where $d\iota_1$ maps $T_{(p,0)} D^{n-k}$ to itself. Write down a map interpolating between ι_1 and a bundle map, and use the inverse function theorem to show it's an isotopy.)

4. An embedded S^{k-1} in S^{n-1} is said to be unknotted if it bounds an embedded disk. Let M be a manifold obtained by attaching $\mathcal{H}_n(k)$ to $\mathcal{H}_n(0)$. If the attaching sphere of the first handle is unknotted, show that M is diffeomorphic to the disk bundle of an $n-k$ dimensional vector bundle over S^k. When $n = 4$ and $k = 2$, show that the set of such vector bundles can naturally be identified with $\mathbb{Z}$. Are any of the associated disk bundles diffeomorphic to each other?

5. Let N^n be a handlebody obtained by starting with a single 0-handle and adding g 1-handles. Show that if $n > 2$, N is diffeomorphic to either a boundary connected sum of g copies of $S^1 \times D^{n-1}$ or to a boundary connected sum of g copies of $\widetilde{M}$, where $\widetilde{M}$ is the nonorientable D^{n-1} bundle over S^1. What goes wrong with your argument when $n = 2$?

6. Let $H(g)$ be the boundary connected sum of g copies of $S^1 \times D^2$. If N is an orientable three-manifold, show that for some $g \geq 0$, $N \simeq H_1 \cup_\phi H_2$, where $H_1 \simeq H_2 \simeq H(g)$ and $\phi : \partial H_1 \to \partial H_2$ is a diffeomorphism. (Such a decomposition is called a *Heegaard*

splitting of genus g for N.) Show that for every G, there is a manifold N_G which does not admit a Heegaard splitting of any genus $g < G$.

7. Consider the 3-manifold obtained by attaching three 2-handles to $H(4)$, as shown by the figure. The figure shows the surface $\partial H(4)$ (in black) together with the belt circles of the 1-handles (blue circles) and the attaching circles of the 2-handles (red circles). What is $H_*(M)$? Now suppose we attach a fourth 2-handle whose attaching circle is shown in green. By sliding and canceling handles, show that the resulting manifold is diffeomorphic to D^3.

8. Suppose N is a 2-dimensional handlebody obtained by attaching k 1-handles to a 0-handle, and that $\partial N \simeq S^1$.

   (a) The handle decomposition of N gives a natural basis for $H_1(N;\mathbb{Z}/2)$. Describe the intersection pairing on $H_1(N,\mathbb{Z}/2)$ in terms of this basis.

   (b) Let B be a nonsingular bilinear pairing on a $\mathbb{Z}/2$ vector space of dimension k. Show that either $B \simeq \oplus_{i=1}^{k/2} H$, where H is the bilinear pairing with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $H = \oplus_{i=1}^{k} A$, where A is the bilinear pairing with matrix (1).

   (c) By imitating your proof of part (b), show by sliding handles that N is diffeomorphic to either the boundary connected sum of k copies of the Mobius band, or $k/2$ copies of the punctured torus.

   (d) Deduce that a closed smooth surface is diffeomorphic to either a connected sum of tori, or a connected sum of projective planes.

9. Suppose (C,d) is a chain complex, that $C_n = C'_n \oplus A$, $C_{n-1} = C'_{n-1} \oplus A$, and that the component of d_n mapping A to A is the identity map. Let (C'', d'') be the chain complex with $C''_i = 0$ unless, $i = n, n-1$, $C''_n \simeq C''_{n-1} \simeq A$, and $d''_n : A \to A$ by the identity map. Show that $(C,d) \simeq (C',d') \oplus (C'',d'')$, where $C'_i = C_i$ for $i \neq n, n-1$, $d'_i = d_i$ for $i \neq n-1, n, n+1$, and d'_{n+1} is the composition of d_{n+1} with the projection onto C'_n. What is d'_n? How is this operation related to handle cancellation?

10. Say that an oriented manifold M^n satisfies strong Poincare duality if the cup product pairing $H^k(M;\mathbb{F}) \times H^{n-k}(M,\partial M;\mathbb{F}) \to \mathbb{F}$ given by $(a,b) \mapsto \langle a \cup b, [M,\partial M]\rangle$ is nonsingular for any field $\mathbb{F}$. Given that closed manifolds satisfy strong Poincare duality and that $H^k(M;\mathbb{F}) \simeq H_{n-k}(M,\partial M;\mathbb{F})$ for any compact manifold with boundary, show that compact manifolds satisfy strong Poincare duality.

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