**Morse Theory** **Lent 2016**

# Example Sheet 1

1. Check carefully that $\phi_{b-a}(M_b) = M_a$, where $\phi_t$ is the flow of $\mathbf{V} = -\alpha\nabla f$ considered in the proof of the theorem from the first lecture.

2. Show that the set of Morse functions on a closed manifold $M$ is open with respect to the topology on $C^\infty(M)$ induced by the $C^2$ metric.

3. Let $f : M \to \mathbb{R}$ be a Morse function, and view $s_1 = df$ as a section of $T^*M$. Let $s_0 \in \Gamma(T^*M)$ denote the zero section, and let $x \in M$ be a critical point of $f$. Show that the local intersection number $s_0 \cdot s_1|_{\tilde{x}}$ is given by $(-1)^{\text{ind}_x f}$. If $M$ is orientable, deduce that the Euler class of $T^*M$ is $\chi(M)[M]^*$, where $[M]^* \in H^n(M)$ is the dual fundamental class of $M$.

4. Suppose $M, N$ are submanifolds of $\mathbb{R}^n$. For $\mathbf{v} \in \mathbb{R}^n$, let $M + \mathbf{v}$ be the image of $M$ under translation by $\mathbf{v}$. Show that for almost every $\mathbf{v} \in \mathbb{R}^n$, $M + \mathbf{v}$ is transverse to $N$.

5. Let $N^n$ be a compact manifold with boundary, and suppose $\iota : \partial_a \mathcal{H}_n(k) \to \partial N$ be an embedding, where $\mathcal{H}_n(k) = D^k \times D^{n-k}$. Let $C = D^k \times 0$ be the core of $\mathcal{H}_n(k)$ and let $i$ be the restriction of $\iota$ to $\partial C$. Show that $N \cup_\iota \mathcal{H}_n(k)$ deformation retracts to $N \cup_i C$. Deduce that a closed manifold is homotopy equivalent to a finite cell complex.

6. Suppose that the set of critical points of $f : M^n \to \mathbb{R}$ is a submanifold $N^k \subset M$, and let $\nu \subset TM$ be the normal bundle to $N$. We say that $f$ is *Morse-Bott* if for all $x \in N$, $T_x N$ is the null space of the Hessian $H_x(f)$. If $f$ is Morse-Bott and $g : N \to \mathbb{R}$ is Morse, show there is a Morse function $h : M \to \mathbb{R}$ whose critical point set is equal to the critical point set of $g$.

7. Find a map $f : T^2 \to \mathbb{R}$ with only 3 critical points.

8. Suppose $M$ is a complex manifold, and that $F : M \to \mathbb{C}$ is a holomorphic map. We say that $p \in M$ is a complex critical point of $F$ if $\frac{\partial F}{\partial z_i}|_p = 0$ for $1 \leq i \leq n$, where $(z_1, \ldots, z_n)$ are local coordinates on $M$ near $p$. We say $p$ is nondegenerate if $\det(\frac{\partial^2 F}{\partial z_i \partial z_j}|_p) \neq 0$. By considering power series expansions, show that if $p$ is a nondegenerate complex critical point of $F$, then there are local coordinates on $M$ near $p$ with respect to which $F(z_1, \ldots, z_n) = f(0) + \sum_{i=1}^n z_i^2$.

9. Let $F$ be as in the previous problem, and define $f : M \to \mathbb{R}$ defined by $f(z) = |F(z)|^2$. If $p$ is a critical point of $f$, show that either $F(p) = 0$ or $p$ is a

complex critical point of $F$. In the latter case, show that if $p$ is a nondegenerate critical point of $F$, then $\text{ind}_p f = n$.

10. If $V \subset \mathbb{C}^m$ is a smooth affine variety of complex dimension $n$ (*i.e* $V$ is the set of solutions to a finite set of polynomial equations and is also a embedded complex submanifold of $\mathbb{C}^m$), show that $V$ is homotopy equivalent to a finite cell complex of dimension $n$.

J.Rasmussen@dpmms.cam.ac.uk