Path and Homotopy Lifting

We fix the disastrous gap between the statement of Lemma 2 on Monday and it's statement on Wednesday. As a bonus, we give an alternate proofs of path and homotopy lifting that illustrate how similar they are. Throughout, suppose $p : \widehat{X} \to X$ is a covering map.

Definition 1. *$f : Z \to X$ has the unique lifting property (ULP) at z in Z if for any $\widehat{x} \in \widehat{X}$ with $p(\widehat{x}) = f(z)$ there is a unique $\widehat{f} : (Z, z) \to (\widehat{X}, \widehat{x})$ with $p \circ \widehat{f} = f$.*

Lemma 2. *Suppose $U \subset X$ is open and evenly covered by p, and that Z is connected. If $f : Z \to \widehat{X}$ has $\operatorname{im} f \subset U$, then f has the unique lifting property at any $z \in Z$.*

Proof. Suppose $p(\widehat{x}) = f(z)$. U is evenly covered, so $p^{-1}(U) \cong \coprod_{\alpha \in A} U_\alpha$, and $p_\alpha = p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism. So there is some $\alpha_0 \in A$ for which $\widehat{x} \in U_{\alpha_0}$.

We define $\widehat{f} = i_{\alpha_0} \circ p_{\alpha_0}^{-1} \circ f$, where $\iota_{\alpha_0} : U_{\alpha_0} \to X$ is the inclusion. This is a composition of continuous functions, hence continuous, and it satisfies $\widehat{f}(z) = p_{\alpha_0}^{-1}(f(z)) = \widehat{x}$. Since $\operatorname{im} \widehat{f} \subset U_{\alpha_0}$, we have $p \circ \widehat{f}(z) = p|_{U_{\alpha_0}} \circ p_{\alpha_0}^{-1} \circ f = f$. Thus $\widehat{f}$ is the desired lift.

For uniqueness, observe that $A = U_{\alpha_0}$ and $B = \bigcup_{\alpha \neq \alpha_0} U_\alpha$ are disjoint open subsets of $p^{-1}(U)$. $A \cup B = p^{-1}(U)$, so A and B disconnect $p^{-1}(U)$. Suppose $\tilde{f} : (Z, z) \to (\widehat{X}, \widehat{x})$ is a lift of f. Since Z is connected and $\tilde{f}(z) = \widehat{x}$, we must have $\tilde{f}(Z) \subset A = U_{\alpha_0}$. Since $\tilde{f}$ is a lift, we have $f = p \circ \tilde{f} = p_{\alpha_0} \circ \tilde{f}$. It follows that $\tilde{f}(z') = p_{\alpha_0}^{-1}(f(z')) = \widehat{f}(z')$, for all $z' \in Z$, i.e. $\tilde{f} = \widehat{f}$.

□

Lemma 3. *Suppose $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are closed in Z, that $z_1 \in Z_1$ and that $z_2 \in Z_1 \cap Z_2$. Given $f : Z \to X$, let $f_i = f|_{Z_i}$, and $g = f|_{Z_1 \cap Z_2}$. If f_i has ULP for z_i $(i = 1, 2)$ and g has ULP for z_2, then f has ULP for z_1.*

Proof. Suppose $p(\widehat{x}) = f(z)$. By ULP for f_1, there is a unique lift $\widehat{f}_1$ of f_1 with $\widehat{f}_1(z_1) = \widehat{x}$. Then $p(\widehat{f}_1(z_2)) = f(z_2)$. By ULP for f_2, there is unique lift $\widehat{f}_2$ of f_2 with $\widehat{f}_2(z_2) = \widehat{f}_1(z_2)$. Let $\widehat{g}_i = \widehat{f}_i|_{Z_1 \cap Z_2}$. Then $p \circ \widehat{g}_i = g$ and $\widehat{g}_1(z_2) = \widehat{g}_2(z_2)$. Since g has ULP for z_2, $\widehat{g}_1 = \widehat{g}_2$. Hence we can define $\widehat{f} : Z \to \widehat{X}$ by $\widehat{f}(z) = \widehat{f}_1(z)$ if $z \in Z_1$ and $\widehat{f}(z) = \widehat{f}_2(z)$ if $z \in Z_2$. $\widehat{f}$ is continuous by the gluing lemma and is clearly a lift of f. This proves existence.

For uniqueness, suppose $\tilde{f}$ is a lift of f with $\tilde{f}(z_1) = \widehat{x}$. Let $\tilde{f}_i = \tilde{f}|_{Z_i}$. Then $\tilde{f}_i$ is a lift of f_i and $\tilde{f}_1(z_1) = \widehat{x}$. By ULP for f_1, $\tilde{f}_1 = \widehat{f}_1$. Hence $\tilde{f}_2(z_2) = \tilde{f}_1(z_2) = \widehat{f}_1(z_2)$. By ULP for f_2, $\tilde{f}_2 = \widehat{f}_2$. Hence $\tilde{f} = \widehat{f}$. □

Lemma 4. *Suppose Z is a compact metric space and $f : Z \to X$. Then there exists $\delta > 0$ such that for every $z \in Z$ there is an open $U_z \subset X$ which is evenly covered by p and for which $f(B_\delta(z)) \subset U_z$.*

Proof. Since p is a covering map, each $x \in X$ has an open neighborhood U_z which is evenly covered by p. Then $\{U_x \,|\, x \in X\}$ is an open cover of X. Since f is continuous, $\{f^{-1}(U_x) \,|\, x \in X\}$ is an open cover of X. By the Lebesgue covering lemma, we can find $\delta > 0$ such that for each $z \in Z$ there is some $x(z) \in X$ with $B_\delta(z) \subset f^{-1}(U_{x(z)})$. Setting $U_z = U_{x(z)}$ gives the statement. □

Theorem. *(Path Lifting) If $f : I \to X$, then f has ULP for 0.*

Proof. I is compact, so choose δ as in Lemma 4 and $n \in \mathbb{Z}$ such that $0 < \frac{1}{n} < \delta$. Let $a_i = \frac{i}{n}$ and let $A_i = [a_i, a_{i+1}]$. Then $f(A_i) \subset f(B_\delta(a_i)) \subset U_{a_i}$, where U_{a_i} is open and evenly covered. A_i is connected, so by Lemma 2, $f|_{A_i}$ has ULP for a_i.$(*)$

For $0 \leq k < n$, let $B_k = \bigcup_{i=0}^{k} A_i$. Then $A_{k+1} \cap B_k = \{a_{k+1}\}$ is connected and contained in A_{k+1}, so by Lemma 2 $f|_{A_{k+1} \cap B_k}$ has the ULP for a_i. $(**)$.

We prove by induction on k that $f|_{B_k}$ has the ULP for 0. When $k = 0$, $B_0 = A_0$, and $f|_{A_0}$ has ULP for 0 by $(*)$. Now suppose the statement holds for k. Now $B_{k+1} = Z_1 \cup Z_2$, where $Z_1 = B_k$ and $Z_2 = A_{k+1}$. With notation as in Lemma 2, f_1 has ULP for 0 by the induction hypothesis, f_2 has ULP for a_{k+1} by $(*)$, and g has ULP for a_{k+1} by $(**)$. By Lemma 3, $f|_{B_{k+1}}$ has ULP for 0. By induction the statement holds for all $k < n$. Taking $k = n - 1$ gives the statement of the theorem. □

Theorem. *(Homotopy Lifting) If $f : I \times I \to X$, then f has ULP for* $(0, 0)$.

Proof. $I \times I$ is compact, so choose δ as in Lemma 4 and $n \in \mathbb{Z}$ such that $0 < \frac{\sqrt{2}}{n} < \delta$. For $0 \leq i, j < n$, let $a_{i+nj} = (\frac{i}{n}, \frac{j}{n})$ and let A_k be the square of side $\frac{1}{n}$ whose lower left corner is at A_i. Then $f(A_i) \subset f(B_\delta(a_i)) \subset U_{a_i}$, where U_{a_i} is open and evenly covered. The rest of the argument now proceeds exactly as in the proof of path lifting, except the check that $A_{k+1} \cap B_k$ is connected is a little less trivial. □