KNOTS AND 4-MANIFOLDS **LENT 2012**

EXAMPLE SHEET 4

1. Use the skein relation for the Kauffman bracket to compute $\overline{V}(T(2,n))$.

2. Show that $\overline{V}(K_1 \# K_2) = \overline{V}(K_1)\overline{V}(K_2)$.

3. Show that $|\overline{V}_K(-1)| = \det K$.

4. Let D be a connected knot diagram. A *checkerboard coloring* of D is a coloring of the regions of D either black or white, such that the two regions on either side of an edge have opposite colors. The *black graph* associated to such a coloring is the planar graph with one vertex in each black region and one edge for each crossing in D. (The edge joins the two black vertices adjacent to the crossing.) Let T be the set of maximal trees in the black graph. If K is represented by the diagram D, use the Kauffman skein relation to show that
$$V(K) = \sum_{\tau \in T} (-1)^{\sigma(\tau)} q^{n(\tau)}$$
where $\sigma(\tau), n(\tau) \in \mathbb{Z}$. If D is alternating, show that $\sigma(\tau) - n(\tau)$ has the same value mod 2 for all $\tau \in T$. Conclude that the number of maximal trees in the black graph is $\det K$.

5. Show that there are only finitely many alternating knots with a given determinant.

6. Let H be the Hopf link, oriented so the linking number is 1. Compute $Kh(H)$ directly from the definition.

7. Let E be the set of edges of an n-dimensional cube. Show that there is a map $\sigma : E \to \{\pm 1\}$ such that every two dimensional face of the cube has an odd number of edges with $\sigma(e) = -1$. Show that if σ_1 and σ_2 are two such sign assignments, the resulting chain complexes (as in the definition of CKh) are isomorphic.

8. Let D be a knot diagram, and fix an edge e of D. Let C_- be the subset of $CKh(D)$ generated by those Kauffman states for which the circle containing e is labeled with an X. Show that C_- is a subcomplex of $CKh(D)$, and that the quotient complex $CKh(D)/C_-$ is isomorphic to C_-. Show that $\chi(C_-) = \overline{V}(K)$.

9. A knot K is *positive* if it can be represented by a diagram D all of whose crossings are positive. Show that if K is a positive knot, then $s(K) = 2g(\Sigma)$ where Σ is a Seifert

surface obtained by applying Seifert's algorithm to a positive diagram of K. Conclude that $g_*(K) = g(K) = g(\Sigma)$.

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