KNOTS AND 4-MANIFOLDS LENT 2012

EXAMPLE SHEET 3

1. If K is a knot in S^3, let K_0 be the manifold obtained by 0 surgery on K. (In other words, K_0 is the boundary of the four-manifold obtained by attaching a 0-framed 2-handle to B^4 along K.) Show that $\Delta(K_0) = \Delta(K)$. Conclude that if $\Delta_K(t) \neq 1$, then $K_0 \neq S^1 \times S^2$. (In fact, $K_0 = S^1 \times S^2$ implies K is the unknot.)

2. Show that the knot shown in Figure 1 below has $\Delta_K(t) = 1$. This knot is the *Whitehead double* of the trefoil. Can you generalize to other knots?

3. A knot $K \subset S^3$ is *fibred* if we can write $S^3 - K = \Sigma \times [0,1]/\sim$, where Σ is a surface with one boundary component and $(x, 0) \sim (\phi(x), 1)$ for some diffeomorphism $\phi : \Sigma \to \Sigma$. Show that $\Delta_K(t) \sim \det(tI - \phi_*)$, where $\phi_* : H_1(\Sigma) \to H_1(\Sigma)$. Conclude that if K is fibred, then $\Delta_K(t)$ is monic and Σ is a minimal genus Seifert surface for K.

4. Let $K_{p/q}$ be the rational knot associated to the fraction p/q. Using the bridge diagram described in class, show that $\pi_1(S^3 - K_{p/q})$ has a presentation of the form

$$\langle a, b \,|\, a^{f(0)} b^{f(1)} \dots b^{f(2p-1)} = 1 \rangle$$

where $f(n) = (-1)^{\lfloor \frac{nq}{p} \rfloor}$.

5. Show that if $q \cong q'$ mod p or $q \cong q'$ mod p, then $K_{p/q} = K_{p/q'}$, and that if $q \cong -q'$ mod p, then $K_{p/q'}$ and $K_{p/q}$ are mirrors. Make a list of all rational knots with ≤ 6 crossings. (Up to mirrors, you should find there are seven.) Compute their Alexander polynomials.

6. Let K be a knot of genus 1. Show that $\sigma(K) \neq 0$ if and only if the leading coefficient of $\Delta_K(t)$ is positive. (As usual, $\Delta_K(t)$ should be normalized so $\Delta_K(1) = 1$.)

7. Find a Seifert matrix for the left-handed $(2, n)$ torus knot shown in Figure 2. Show that the signature of this knot is $n - 1$.

8. Suppose that K_+ is a knot obtained from K_- by changing a negative crossing to a positive one. Show that $\sigma(K_+) \leq \sigma(K_-) \leq \sigma(K_+) + 2$. Show further that $\sigma(K_+) = \sigma(K_-)$ if and only if $\Delta_{K_+}(-1)$ and $\Delta_{K_-}(-1)$ have the same sign. Use this fact to compute the signatures of the knots you listed in problem 5.

9. With notation as above, show that $|g_*(K_+) - g_*(K_-)| \leq 1$. Conclude that if K can be unknotted by changing n crossings in a diagram representing K, then $g_*(K) \leq n$.

10. Suppose that $Y^3 = \partial W^4$, where $H_*(Y;\mathbb{Q}) \simeq H_*(S^3;\mathbb{Q})$ and $H_*(W;\mathbb{Q}) \simeq H_*(B^4;\mathbb{Q})$. Show that $|H_1(Y)| = |H_1(W)|^2$. Conclude that if $K \subset S^3$ is slice, then $\det K$ is a perfect square.

11. Using the previous three problems, compute $g_*(K)$ for the knots you listed in problem 5. (You should find exactly one slice knot.)

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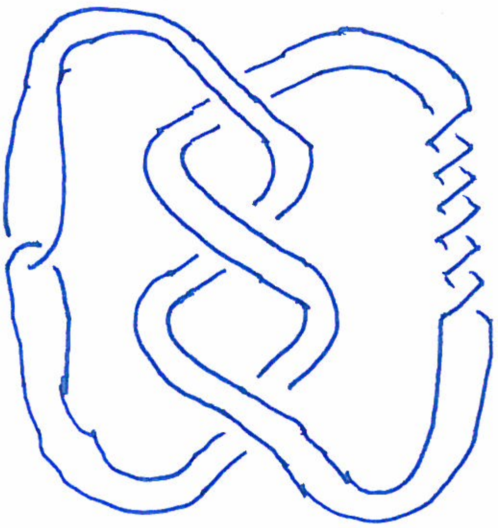
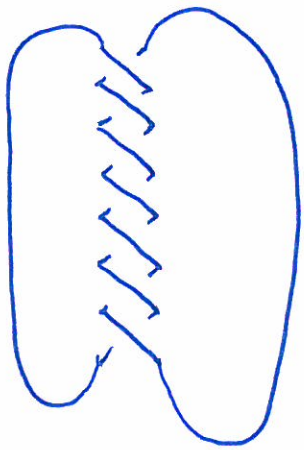


Figure 1



T(2,7)

Figure 2