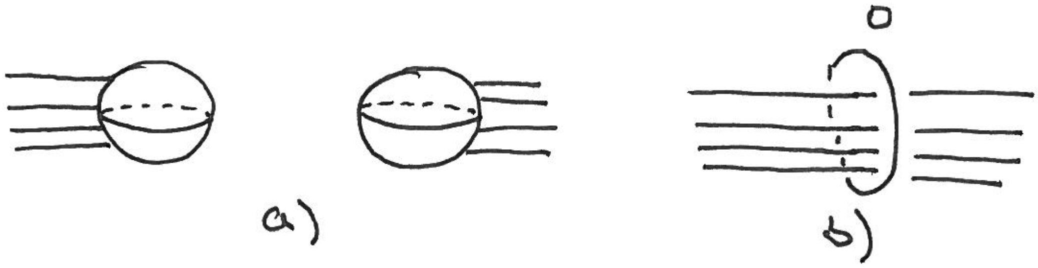
KNOTS AND 4-MANIFOLDS LENT 2012

EXAMPLE SHEET 2

1. Suppose M is a four-dimensional handlebody composed of 0, 1, and 2-handles, and that the corresponding Kirby diagram contains a region like that shown in Figure a). (The attaching circles of several two-handles pass over a 1-handle.) Let M' be the handlebody whose Kirby diagram obtained by replacing the region in Figure a) with that in Figure b). Show that $\partial M \simeq \partial M'$.



2. Given that every connected orientable Y^3 bounds an orientable 4-manifold, show that every such Y^3 is obtained by integral surgery on some link in S^3. (*Hint:* Use problem 1.)

3. a) Prove that any finitely-presented group is the fundamental group of a compact four-manifold with boundary. b) Show that any finitely presented group is the fundamental group of a closed four-manifold. (*Hint:* let M be the manifold from a). Starting from a Kirby diagram of M, attach a zero-framed 2-handle along the meridian of each 2-handle of M to obtain a new manifold M'. Show that $\partial M'$ is $\#^n S^1 \times S^2$, where n is the number of 1-handles in M.)

4. If K_1 and K_2 are knots, show that $\Delta(K_1 \# K_2) = \Delta(K_1)\Delta(K_2)$. If L is the link obtained by taking the disjoint union of K_1 and K_2, show that the multivariable Alexander polynomial of L is 0.

5. Let T be the standard unknotted torus in S^3 (*i.e.* the boundary of a tubular neighborhood of the unknot). The (p, q) *torus knot* is the simple closed curve on T representing the class $p\ell + qm$ in $H_1(T^2)$, where p and q are relatively prime. Use the Seifert-Van Kampen theorem to show that $\pi_1(S^3 - T(p, q)) = \langle a, b \mid a^p = b^q \rangle$. Compute $\Delta(T(p, q))$.

6. Compute $\Delta(K)$ for the knots shown in Figure 1 below using a) Fox calculus, b) Seifert matrices, and c) the skein relation.

7. Compute the multivariable Alexander polynomial of each of the links shown in Figure 2 below. In each case, find embedded surfaces representing the Poincare duals to $m_1, m_2, m_1 + m_2$ and $m_1 - m_2$, where m_1 and m_2 are meridians of the two components. Do your surfaces satisfy $\chi(S) = 2 \max_{b \in B} b \cdot [S]$?

8. Compute the multivariable Alexander polynomial of $S^1 \times \Sigma_g$, where Σ_g is a closed surface of genus g.

9. Let K be the trefoil knot in S^3, and let Y_k be the k-fold cyclic cover of $S^3 - K$; that is, Y_k is the regular covering space of $S^3 - K$ corresponding to the homomorphism $\pi_1(S^3 - K) \to H_1(S^3 - K) \to \mathbb{Z}/k$, where the second arrow is the map $\mathbb{Z} \to \mathbb{Z}/k$ which takes $1 \in \mathbb{Z}$ to $1 \in \mathbb{Z}/k$. Show that $Y_{k+6} \simeq Y_k$. Compute $H_1(Y_k)$ for all $n > 0$.

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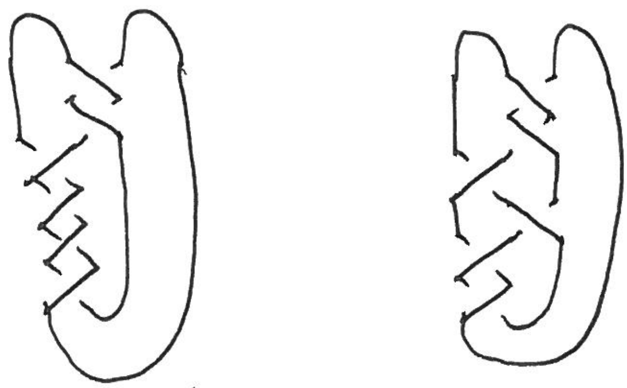

Figure 1

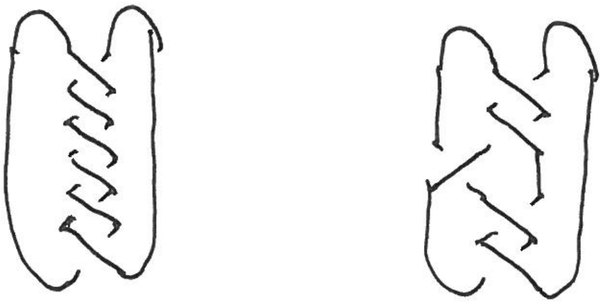

Figure 2