KNOTS AND 4-MANIFOLDS **LENT 2012**

EXAMPLE SHEET 1

1. Suppose W^m is a cobordism from M_1 to M_2. Use handle decompositions to prove that $H_i(W, M_1) \cong H^{m-i}(W, M_2)$.

2. (a) Let Q be a nondegenerate quadratic form on a real vector space V. Suppose that V has a subspace L whose dimension is half that of V and that $Q(x, y) = 0$ for all $x, y \in L$. Show that $\sigma(Q) = 0$.

   (b) Suppose that M is a $4n$-manifold which bounds a $(4n + 1)$-manifold W. Use the long exact sequence of the pair (W, M) to show that $\sigma(M) = 0$. Conclude that $\mathbb{CP}^{2n}$ does not bound an orientable $(4n + 1)$-manifold.

3. Suppose that $M^m = B^m \cup_f B^m$, where $f : S^{m-1} \to S^{m-1}$ is an orientation reversing diffeomorphism. Show that M is homeomorphic to S^m. (Hint: B^m is the cone on S^{m-1}.)

4. Show that any embedding of S^1 into an orientable manifold has trivial normal bundle. Show that an embedding of S^2 into an orientable three-manifold has trivial normal bundle.

5. Prove that a closed orientable surface is diffeomorphic to either a connected sum of T^2's (if it is orientable), or $\mathbb{RP}^2$'s (if it is not.) (Hint: first show the $\mathbb{Z}/2$ intersection form can be reduced to a standard form. Then slide handles to match.)

6. Draw a Heegaard diagram representing T^3. Check that the result has $\pi_1 \cong \mathbb{Z}^3$.

7. Draw Heegaard diagrams representing manifolds Y with torus boundary which give the following presentations of $\pi_1(Y)$:

   (a) $\pi_1(Y) = \langle a, b, c \,|\, bac^{-1}a^{-1} = acb^{-1}c^{-1} = 1\rangle$.

   (b) $\pi_1(Y) = \langle a, b, c, d \,|\, bc^{-1}a = c^{-1}bd^{-1}a = d^{-1}ba = 1\rangle$.

   Use handleslides and cancellations to transform your diagrams into either the genus two diagram we drew in class with $\pi_1(Y) = \langle x, y \,|\, xyxy^{-1}x^{-1}y^{-1} = 1\rangle$, or its mirror image. (Hint: first show the groups are isomorphic.)

8. Draw Kirby diagrams representing $(S^2 \times S^2)\#\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. Slide handles to show that they are diffeomorphic. (Hint: first find the isomorphism of intersection forms.)

9. Consider a Kirby diagram obtained by attaching n 2-handles to a 0-handle with attaching circles K_i. Attach an additional n 2-handles along small circles linking the K_i. The new handles should all have framing zero. Prove that the resulting manifold is diffeomorphic to the complement of a ball in a connected sum of $S^2 \times S^2$'s, $\mathbb{CP}^2$'s, and $\overline{\mathbb{CP}}^2$'s.

10. Find the intersection forms on the four-manifolds represented by the Kirby diagrams below. (Each diagram is the union of a 0-handle and some 2-handles.) In each case, compute the signature and H_1 of the boundary.

J.Rasmussen@dpmms.cam.ac.uk