1. Isom(D)

Let (D, g^D) be the unit disk model of the hyperbolic plane. Recall that

$$G^D = \left\{ \phi \in \text{Mob} \,\middle|\, \phi(z) = e^{i\theta} \frac{z-a}{\bar{a}z - 1}, \theta \in \mathbb{R}, |a| < 1 \right\}$$

is contained in Isom(D). Let $c : D \to D$, $c(z) = \overline{z}$ be reflection across the real axis. It is easy to check that $c \in Isom(D)$ as well, but $c \notin$ Mob, since it is orientation reversing.

Theorem 1. *If $\phi \in Isom(D)$, then either $\phi \in G^D$ or $\phi = \tilde{\phi} \circ c$ for some $\tilde{\phi} \in G^D$.*

Lemma 2. *Suppose $\phi \in Isom(D)$, $\phi(0) = 0$ and $d\phi|_0 = I$. Then $\phi = 1_D$.*

Proof. ϕ preserves lengths of paths, so if γ is the shortest path from $\mathbf{p}$ to $\mathbf{q}$, $\phi \circ \gamma$ is the shortest path from $\phi(\mathbf{p})$ to $\phi(\mathbf{q})$. Since the unique shortest path between two points is the hyperbolic line segment joining them, it follows that ϕ must send hyperbolic rays to hyperbolic rays. Since there is a unique hyperbolic ray passing through a given point in a given direction, ϕ takes the ray starting at $\mathbf{p}$ with direction vector $\mathbf{a}$ to the ray starting at $\phi(\mathbf{p})$ with direction $d\phi|_{\mathbf{p}}(\mathbf{a})$.

Setting $\mathbf{p} = 0$, we see that ϕ takes each Euclidean ray starting at the origin to itself. Moreover, ϕ preserves the hyperbolic distance. Since the hyperbolic distance from the origin increases monotonically with the Euclidean distance, it follows that ϕ is the identity on each ray, and thus on all of D. □

Lemma 3. *If $\phi \in Isom(D)$ and $\phi(0) = 0$, then either $\phi(z) = e^{i\theta}z$ or $\phi(z) = e^{i\theta}\overline{z}$ ($\theta \in \mathbb{R}$).*

Proof. Since ϕ is an isometry $g_0^D(\mathbf{a}, \mathbf{b}) = g_0^D(d\phi|_0(\mathbf{a}), d\phi|_0(\mathbf{b}))$. Noting that g_0^D is the ordinary Euclidean inner product, we conclude that $d\phi|_0 \in O(2)$. Thus either $d\phi|_0 = r_\theta$ or $d\phi|_0 = r_\theta \circ c$, where $r_\theta(z) = e^{i\theta}z$. In the first case, note that $r_\theta \in$ Isom(D), so $\psi = \phi \circ r_{-\theta} \in$ Isom(D) has $\psi(0) = 0$ and

$$d\psi|_0 = d\phi|_0 \circ dr_{-\theta}|_0 = r_\theta \circ r_{-\theta} = I.$$

(Here we have the fact that the derivative of a linear map L is L.) It follows that $\psi = 1_D$, so $\phi = r_\theta$. The second case is similar. □

Proof. (Of Theorem 1) Let $\mathbf{p} = \phi(0)$. Choose $\psi \in G^D$ with $\psi(\mathbf{p}) = 0$. Then $\psi \circ \phi \in$ Isom(D), $\psi \circ \phi(0) = 0$, so either $\psi \circ \phi = r_\theta$ or $\psi \circ \phi = r_\theta \circ c$. Thus either $\phi = \psi^{-1} \circ r_\theta$ or $\phi = \psi^{-1} \circ r_\theta \circ c$. Since $r_\theta, \psi^{-1} \in G^D$ the result follows. □

If (H, g^H) is the upper half-plane model of hyperbolic space, recall that $G^H = PSL_2(\mathbb{R}) \subset$ Isom(H).

Corollary 4. *If $\phi \in Isom(H)$, then either $\phi \in G^H$ or $\phi = \tilde{\phi} \circ r$, where $r(z) = -\overline{z}$ is reflection across the imaginary axis, and $\tilde{\phi} \in G^H$.*

Proof. Let $\phi_0 : H \to D$ be given by $\phi_0(z) = \frac{z-i}{z+i}$. Then $\phi_0 \circ \phi \circ \phi_0^{-1} \in$ Isom(D), so either $\phi = \phi_0^{-1} \circ \psi \circ \phi_0$ or $\phi = \phi_0^{-1} \circ \psi \circ c \circ \phi_0$ where $\phi \in G^D$. In the first case, the result is immediate from the fact that $G^H = \phi_0^{-1} G^D \phi_0$, while in the second it follows from this and the fact that $c \circ \phi_0 = \phi_0 \circ r$, which is easily checked by direct calculation. □

2. Isometries and the extended Möbius group

Extend complex conjugation to a map $c : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by setting $c(\infty) = \infty$. If ϕ_A is the Mobius transformation defined by the matrix $A \in GL_2(\mathbb{C})$, it is easy to see that

$$\phi_A \circ c = c \circ \phi_{\overline{A}}.$$

Definition 5. *The* extended Möbius group *is*

$$\overline{\text{Mob}} = \{\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty \,|\, \phi \in \text{Mob} \textit{ or } \phi \circ c \in \text{Mob}\}$$

Since $c^2 = 1$, the second condition in the definition is equivalent to saying that $\phi = \psi \circ c$, where $\psi \in \text{Mob}$. It follows from the equation above that $\overline{\text{Mob}}$ is closed under composition, and thus forms a group. It contains the usual group of Möbius transformations as an index 2 subgroup. Elements of Mob are *orientation preserving*, while elements of $\overline{\text{Mob}}$ which are not contained in Mob are *orientation reversing.*

Definition 6. *If $C \subset \mathbb{C}_\infty$ is a Euclidean line/circle, reflection in C is the extended Mobius transformation defined by $R_C = \psi^{-1} \circ c \circ \psi$, where $\psi \in \text{Mob}$ has $\psi(C) = \mathbb{R} \cup \{\infty\}$.*

We should check that the definition does not depend on the choice of ψ. If $\psi'(C) = \mathbb{R} \cup \{\infty\}$, then $\psi' \circ \psi^{-1}(\mathbb{R}) = \mathbb{R}$, so $\psi' \circ \psi^{-1} = \phi_A$, where $A \in GL_2(\mathbb{R})$. Then

$$\psi'^{-1} \circ c \circ \psi' = \psi^{-1} \circ \phi_A^{-1} \circ c \circ \phi_A \circ \psi = \psi^{-1} \circ c \circ \psi$$

so R_C is well defined.
Example: If C is the unit circle, then $R_C = \phi_0 \circ c \circ \phi_0^{-1}$, so

$$\begin{aligned} R_C(z) &= \phi_0\left(-i\frac{\overline{z}+1}{\overline{z}-1}\right) \\ &= \frac{-i\frac{\overline{z}+1}{\overline{z}-1} - i}{-i\frac{\overline{z}+1}{\overline{z}-1} + i} = \frac{1}{\overline{z}} \end{aligned}$$

More generally, if C_r is a circle of radius r centered at 0, then $R_{C_r} = \psi_r \circ R_{C_1} \circ \psi_{1/r}$, where $\psi_a(z) = az$. Thus $R_{C_r}(z) = r^2/\overline{z}$.

Proposition 7. $\overline{\text{Mob}}$ *is generated by reflections*

Proof. It suffices to check that the maps 1) $z \to z + b$, 2) $z \to az$, and 3) $z \to 1/z$ are compositions of reflections, since these maps generate Mob. Map 1 is $R_{L_1} \circ R_{L_2}$ where L_1 and L_2 are two Euclidean lines perpendicular to b and separated by a distance $|b/2|$. For Map 2), multiplication by $a \in \mathbb{R}$ is $R_{C_1} \circ R_{C_2}$, where C_2 is the unit circle and C_1 is a circle of radius $\sqrt{a}$ centered at 0, while multiplication by $e^{i\theta}$ is $R_{L_1} \circ R_{L_2}$, where L_1 and L_2 are two line which intersect in an angle $\theta/2$ at 0. Finally, map 3) is the composition of reflection in the unit circle with reflection in $\mathbb{R}$. □

The groups $\text{Isom}(S^2)$, $\text{Isom}(\mathbb{R}^2)$ and $\text{Isom}(D)$ may all be viewed as subgroups of the extended Möbius group, corresponding to the extension of the following subgroups of Mob by c.

$$\text{Isom}^+(S^2) = \left\{\phi_A \,\middle|\, A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \det A = 1\right\}$$

$$\text{Isom}^+(\mathbb{R}^2) = \left\{\phi_A \,\middle|\, A = \begin{pmatrix} \alpha & \beta \\ 0 & \overline{\alpha} \end{pmatrix}, \det A = 1\right\}$$

$$\text{Isom}^+(D) = \left\{\phi_A \,\middle|\, A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \det A = 1\right\}$$