Hyperbolic Geometry

Models

- Hyperboloid model: $S = \{(x, y, z) \,|\, x^2 + y^2 - z^2 = -1, z < 0\}$, metric induced by the Minkowski metric $dx^2 + dy^2 - dz^2$ on $\mathbb{R}^3$. Lines are intersections of S with planes through the origin.
- Unit disk model: $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ with metric
$$g^D = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}.$$
Lines are Euclidean lines/circles perpendicular to ∂D.
- Upper half plane model: $H = \{z \in \mathbb{C} \,|\, \mathrm{Re} z > 0\}$ with metric
$$g^H = \frac{dx^2 + dy^2}{y^2}.$$
Lines are Euclidean lines/circles perpendicular to ∂H.

Lines

- A line is the shortest path between two points.
- Plane separation: the complement of a line is a disconnected topogical space.
- There is a unique line passing through two distinct points.
- Two distinct lines intersect in at most one point.
- Given a point $\mathbf{x}$ and a line L not containing $\mathbf{x}$, there is a unique line passing through $\mathbf{x}$ and perpendicular to L.
- Given a point $\mathbf{x}$ and a line L not containing $\mathbf{x}$, there are infinitely many lines passing through $\mathbf{x}$ which do not intersect L.

Circles

- In either the upper half-plane or the unit disk models, circles are Euclidean circles (but their centers are not the Euclidean centers.)
- A line and a circle intersect in at most two points.
- Two distinct circles intersect in at most two points.
- The perimeter of a circle of radius R is $2\pi \sinh R$.

Isometries

- If F_1 and F_2 are orthogonal frames, there is a unique isometry taking F_1 to F_2.
- An isometry which fixes three non-colinear points is the identity.
- Any isometry can be written as the composition of ≤ 3 reflections.

Triangles

- The sum of the interior angles in ΔABC is $\pi - \mathrm{Area}(ABC)$.
- If A_1, A_2, A_3 and A'_1, A'_2, A'_3 are two sets of non-colinear points with $d(A_i, A_j) = d(A'_i, A'_j)$, then there is a unique $\phi \in \mathrm{Isom}(\mathbb{H})$ with $\phi(A_i) = A'_i$.

- If A_1, A_2, A_3 and A'_1, A'_2, A'_3 are two sets of non-colinear points with $d(A_1, A_j) = d(A'_1, A'_j)$ and $\angle A_2 A_1 A_3 = \angle A'_2 A'_1 A'_3$ then there is a unique $\phi \in \mathrm{Isom}(\mathbb{H}^2)$ with $\phi(A_i) = A'_i$.

Trigonometry

If $\triangle ABC$ has sides a, b, c and opposite angles α, β, γ, then

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$$

$$\cosh a = \cosh b \cosh c - \cos\alpha \sinh b \sinh c$$

$$\cos\alpha = -\cos\beta\cos\gamma + \cosh a \sinh\beta \sinh\gamma$$

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