IB GEOMETRY LENT 2013

EXAMPLE SHEET 3

1. Using the geodesic equations, show directly that the geodesics in the hyperbolic plane are hyperbolic lines parametrized with constant speed. (Hint: first consider vertical lines in the upper half-plane model.)

2. Let S be the cylinder $S = \{(x, y, z) \mid x^2 + y^2 = 1\}$. Prove that S is locally isometric to the Euclidean plane. Show all geodesics on S are spirals of the form $\gamma(t) = (\cos at, \sin at, bt)$ where $a^2 + b^2 = 1$.

3. For $a > 0$, let S be the circular half-cone $\Sigma = \{(x, y, z) \mid z^2 = a(x^2 + y^2), z > 0\}$. Show that Σ minus a ray through the origin is locally isometric to the Euclidean plane. (Hint: identify the edges of a circular sector.) When $a = 3$, give an explicit formula for the geodesics on S and show that no geodesic intersects itself. For $a > 3$ show that there are geodesics which intersect themselves.

4. Let V be the set of smooth functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(t)dt = k$. If $F : V \to \mathbb{R}$ is given by $F(f) = \int_0^1 f(t)^2 dt$, show that the only critical point of F is the constant function $f(t) = k$. Deduce that geodesics have constant speed.

5. Let g^D be the hyperbolic metric on the unit disk. How are geodesic polar coordinates centered at the origin related to usual (Euclidean) polar coordinates on D? Show that with respect to geodesic polar coordinates, the hyperbolic metric takes the form $dr^2 + sinh^2 r\ d\theta^2$. Conclude that at every point of D, the Gauss curvature is -1. What happens if instead of g^D, we use the spherical metric g^S on $\mathbb{C}$?

6. Find an atlas of charts on S^2 for which each chart preserves area, and the transition functions relating charts have derivatives with determinant 1. (Hint: consider the circumscribed cylinder.)

7. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $S \subset \mathbb{R}^3$ be its graph. Show that S is an embedded surface, and that its Gauss curvature at the point $(x, y, F(x, y))$ is the value of

$$\frac{F_{xx}F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}$$

at the point (x, y).

8. Let γ be an embedded curve in the xz-plane given by the parametrization $\gamma(t) = (f(t), 0, g(t))$, where $f(t) > 0$ for all t, and let S be the surface obtained by rotating γ around the z-axis. Show that the Gauss curvature of S is

$$K = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g}}{f(\dot{f}^2 + \dot{g}^2)^2}.$$

If γ is parametrized so as to have unit speed ($\dot{f}^2 + \dot{g}^2 = 1$), show that this reduces to $K = -\ddot{f}/f$.

9. Using the previous question, compute the Gauss curvature of the surfaces given by the equations $x^2 + y^2 - z^2 = 1$ and $x^2 + y^2 - z^2 = -1$. Describe the qualitative properties of the curvature in these cases (sign and behavior near ∞) and explain what you find using pictures of these surfaces.

10. Let T be the torus obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z-axis. Find the Gauss curvature K of T, and identify the points on T where K is positive, negative, and zero. Verify that

$$\int_T K \, dA = 0.$$

11. Let D be an open disc centered at the origin in $\mathbb{R}^2$. Give D a Riemannian metric of the form $(dx^2 + dy^2)/f(r)^2$, where $r = \sqrt{x^2 + y^2}$ and $f(r) > 0$. Show that the curvature of this metric is $K = ff'' - (f')^2 + ff'/r$.

12. Show that the embedded surface given by the equation $x^2 + y^2 + c^2 z^2 = 1$ $(c > 0)$ is homeomorphic to S^2. Deduce from the global Gauss-Bonnet theorem that

$$\int_0^1 (1 + (c^2 - 1)u^2)^{-3/2} du = c^{-1}.$$

Can you verify this formula directly?

---

13. Let S be a compact embedded surface in $\mathbb{R}^3$. By considering the smallest closed ball centered at the origin which contains S, show that the Gauss curvature must be strictly positive at some point of S. Conclude that the locally Euclidean metric on the torus cannot obtained as the first fundamental form of a smoothly embedded torus in $\mathbb{R}^3$.

14. Show that a genus two surface can be obtained by appropriately identifying the sides of a regular octogon. Using problem 10 on example sheet 2, show that the genus two surface admits a Riemannian metric with constant curvature $K = -1$. Explain how to generalize your argument to arbitrary surfaces of genus $g > 1$.

15. Let $\mathbf{p}$ be a point on a surface $S \subset \mathbb{R}^3$, and let $\mathbf{n}$ be normal to S at $\mathbf{p}$. If $\mathbf{v} \in T_p(S)$ let $H_{\mathbf{v}}$ be the plane spanned by $\mathbf{n}$ and $\mathbf{v}$, and let $C_{\mathbf{v}} = S \cap H_{\mathbf{v}}$. Show that $B_{II}(\mathbf{v}, \mathbf{v})$ is the curvature (in the sense of problem 14 on example sheet 1) of $C_{\mathbf{v}}$.

16. Suppose S is a surface of revolution obtained by rotating a curve γ in the xz-plane about the z-axis. Find γ such that the Gauss curvature of S is identically -1.

J.Rasmussen@dpmms.cam.ac.uk