**IB GEOMETRY** **LENT 2013**

**EXAMPLE SHEET 1**

1. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in Euclidean space, and let $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$, where $a, b > 0$. Show that $\angle \mathbf{v}_1\mathbf{v}_2 = \angle \mathbf{v}_1\mathbf{v}_3 + \angle \mathbf{v}_3\mathbf{v}_2$. (Hint: choose an advantageous basis.)

2. Show that the sum of the interior angles in a Euclidean triangle is $\pi$. Why doesn't your argument work on the sphere? Show that the sum of the exterior angles of a convex polygon in the Euclidean plane is $2\pi$.

3. Suppose that $L_1$ and $L_2$ are non-parallel lines in $\mathbb{R}^2$, and that $R_i : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the reflection in the line $L_i$ for $i = 1, 2$. Show that the composition $R_1R_2$ is a rotation. Describe the center and angle of rotation in terms of $L_1$ and $L_2$.

4. Suppose that $H$ is a hyperplane in $\mathbb{R}^n$ defined by the equation $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant $c$. The reflection in $H$ is the map from $\mathbb{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u} - c)\mathbf{u}$. Show this is an isometry. If $\mathbf{p}$ and $\mathbf{q}$ are points of $\mathbb{R}^n$, show that there is an $H$ so that reflection in $H$ maps $\mathbf{p}$ to $\mathbf{q}$.

5. Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^n$. Show that the set of points in $\mathbb{R}^n$ which are equidistant from $\mathbf{x}$ and $\mathbf{y}$ is a hyperplane orthogonal to the line segment $\mathbf{xy}$ and passing through its midpoint. (Hint: after applying an isometry, it suffices to consider the case where $\mathbf{x}$ and $\mathbf{y}$ lie on a coordinate axis.) Deduce that every isometry of $\mathbb{R}^n$ is the product of at most $n + 1$ reflections, and that every isometry of $S^2$ is the product of at most 3 reflections.

6. Show that two distinct Euclidean circles intersect in at most two points. If $A_1, A_2, A_3$ and $B_1, B_2, B_3$ are two sets of non-colinear points in $\mathbb{R}^2$, and $d(A_i, A_j) = d(B_i, B_j)$ for all choices of $i$ and $j$, deduce that there is a unique $\phi \in \text{Isom}(\mathbb{R}^2)$ with $\phi(A_i) = B_i$.

7. Let $G$ be a finite subgroup of $\text{Isom}(\mathbb{R}^n)$. By considering the barycentre (*i.e.* average) of the orbit of the origin under $G$, show that $G$ fixes some point of $\mathbb{R}^n$. If $n = 2$, show that $G$ is either cyclic or *dihedral* (that is $D_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and for $n \geq 3$, $D_{2n}$ is the full symmetry group of a regular $2n$-gon.)

8. Let $\Delta$ be a spherical triangle with sides of length $a, b, c$ and opposite angles $\alpha, \beta, \gamma$. Extend the sides of $\Delta$ to form complete great circles. Show that this divides the sphere into 8 triangles and find the side lengths and angles for each.

9. Find the circumference and area of a circle of radius $r$ on $S^2$.

10. Prove that Mobius transformations of $\mathbb{C}_\infty$ preserve cross ratios. If $u, v, \in \mathbb{C}$ correspond to points $\mathbf{p}, \mathbf{q}$ on $S^2$, and $d$ denotes the angular distance from $\mathbf{p}$ to $\mathbf{q}$ on $S^2$, show that $-\tan^2(d/2)$ is the cross ratio of the points $u, v, -1/\overline{u}, -1/\overline{v}$, taken in an appropriate order.

11. Show that any Möbius transformation $T \neq 1$ on $\mathbb{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of $S^2$ through a nonzero angle has exactly two fixed points

$z_1$ and $z_2 = -1/\bar{z}_1$. If $T$ is a Möbius transformation with two fixed points $z_1$ and $z_2 = -1/\bar{z}_1$, show that either $T$ corresponds to a rotation of $S^2$, or one of the fixed points — say $z_1$ — is an attracting fixed point; that is for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

12. Suppose we have a polygonal decomposition of $S^2$ by convex geodesic polygons, where each polygon is contained in some hemisphere. Denote by $F_n$ the number of faces with precisely $n$ edges, and $V_m$ the number of vertices where precisely $m$ edges meet; show that $\sum_n nF_n = 2E = \sum_m mV_m$.

    Suppose that $V_i = F_i = 0$ for $i < 3$. If in addition $V_3 = 0$ , deduce that $E \geq 2V$. Similarly, if $F_3 = 0$, deduce that $E \geq 2F$. Conclude that $V_3 + F_3 > 0$. Prove the identity
    $$\sum_n (6-n)F_n = 12 + 2\sum_m (m-3)V_m.$$
    Deduce that $3F_3 + 2F_4 + F_5 \geq 12$. The surface of a football is decomposed into spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there?

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13. Suppose that $\phi \in \mathrm{Isom}(\mathbb{R}^2)$. Show that there is either a point $x \in \mathbb{R}^2$ with $\phi(x) = x$ or a line $L$ with $\phi(L) = L$. Conclude that $\phi$ is either (a) a translation, (b) a rotation, (c) a reflection, or (d) a composition $R \circ T$, where $R$ is reflection in a line $L$ and and $T$ is translation by some vector parallel to $L$. How does this relate to problem 5?

14. Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is a smooth curve with $|\gamma'(t)| = 1$. Let $\mathbf{n}(t)$ be the unit normal vector to $\gamma'(t)$, chosen so that $(\gamma'(t), \mathbf{n}(t))$ is a positively oriented basis of $\mathbb{R}^2$. Show that $\gamma''(t) = \kappa(t)\mathbf{n}(t)$ for some $\kappa(t) : [0,1] \to \mathbb{R}$ and that $|\kappa(t)| = 1/R(t)$, where $R(t)$ is the radius of the Euclidean circle which is "maximally tangent" to $\gamma$ at $\gamma(t)$. If $\gamma$ is a smooth simple closed curve given in polar coordinates by $r = r(\theta) > 0$, show that the total curvature $\int \kappa(t)dt = \pm 2\pi$. What does this have to do with problem 2? Give an example of a closed $\gamma$ whose total curvature is 0.

15. A spherical triangle $\Delta = ABC$ has vertices given by unit vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbb{R}^3$, sides of length $a, b, c$, and angles $\alpha, \beta, \gamma$. The *polar triangle* $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$, and $\mathbf{B} \times \mathbf{A}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$, and $\pi - a$, $\pi - b$ and $\pi - c$, respectively. Deduce that
    $$\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha \cos\beta.$$

16. Find $X \subset \mathbb{R}^2$ such that (a) any two points $x, y \in X$ can be joined by a continuous path $\gamma : [0,1] \to X$ and (b) for $x \neq y$ the length of any such path is infinite.

J.Rasmussen@dpmms.cam.ac.uk