Review of Möbius Transformations

The Möbius Group

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere*. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, we define a map $\varphi_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $\varphi(z) = \dfrac{az+b}{cz+d}$. The map φ_A is called a *Möbius transformation.*

Lemma. $\varphi_{AB} = \varphi_A \circ \varphi_B$.

This is easily proved by direct computation. More conceptually, we can argue as follows.

Proof. Consider the map $\pi : \mathbb{C}^2 - \{\mathbf{0}\} \to \mathbb{C}_\infty$ given by $\pi(w_1, w_2) = w_1/w_2$. We define an equivalence relation $\sim$ on $\mathbb{C}^2 - \{\mathbf{0}\}$ by setting $\mathbf{w} \sim \mathbf{v}$ if and only if $\pi(\mathbf{w}) = \pi(\mathbf{v})$. This occurs if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{C}^*$.

Now consider the action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2 - \{\mathbf{0}\}$ defined by matrix multiplication: $A \cdot \mathbf{w} = A\mathbf{v}$. Clearly $A \cdot (\lambda\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A \cdot \mathbf{v})$, so if $\pi(\mathbf{w}) = \pi(\mathbf{v})$, then $\pi(A \cdot \mathbf{w}) = \pi(A \cdot \mathbf{v})$. Thus the action of $GL_2(\mathbb{C})$ on $\mathbb{C} - \{\mathbf{0}\}$ descends to an action on $\mathbb{C}_\infty$. Since $z = \pi(z, 1)$, we see that

$$A \cdot z = \pi(A \cdot (z, 1)) = \pi(az + b, cz + d) = \varphi_A(z).$$

Thus

$$\varphi_{AB}(z) = AB \cdot z = A \cdot (B \cdot z) = \varphi_A(\varphi_B(z)).$$

□

Since φ_I is the identity map on $\mathbb{C}_\infty$, it follows that the set of Mobius transformations forms a group under composition.

Definition. *The Möbius group* $\mathrm{Mob} := \{\varphi_A \mid A \in GL_2(\mathbb{C})\}$.

The map $GL_2(\mathbb{C}) \to \mathrm{Mob}$ given by $A \mapsto \varphi_A$ is a surjective homomorphism. Its kernel K is the set of all 2×2 matrices of the form λI, where $\lambda \in \mathbb{C}^*$, and $\mathrm{Mob} \simeq GL_2(\mathbb{C})/K$. The quotient on the right-hand side is the *projective linear group* $PGL_2(\mathbb{C})$. Now any $A \in GL_2(\mathbb{C})$ can be written as $A = (\lambda I)A'$, where $\lambda^2 = \det A$ and $\det A' = 1$, *i.e.* $A' \in SL_2(\mathbb{C})$. Thus

$$PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/K \simeq SL_2(\mathbb{C})/(K \cap SL_2(\mathbb{C})) = SL_2(\mathbb{C})/\{\pm I\} =: PSL_2(\mathbb{C})$$

To sum up, we have proved

Proposition. $\mathrm{Mob} \simeq PSL_2(\mathbb{C})$.

Properties of Möbius Transformations

Proposition. *Any* $\varphi \in \mathrm{Mob}$ *is conformal,* i.e. *it preserves angles.*

Proof. Mobius transformations are holomorphic, so this follows from the general fact that holomorphic maps are conformal (*c.f.* Example Sheet 1, Problem 13). □

Proposition. *If* z_1, z_2, z_3 *and* w_1, w_2, w_3 *are two sets of three distinct points in* $\mathbb{C}_\infty$*, there is a unique* $\varphi \in \mathrm{Mob}$ *with* $\varphi(z_i) = w_i$*,* $(i = 1, 2, 3)$*.*

Proof. Since Mob is a group acting on $\mathbb{C}_\infty$, it suffices to prove the statement in the case where $(z_1, z_2, z_3) = (0, 1, \infty)$. In order for $\varphi_A(z_i) = w_i$, we must have

$$\frac{b}{d} = w_1 \quad \frac{a}{c} = w_2 \quad \frac{a+c}{b+d} = w_3.$$

In light of the first two equations, the last equation reduces to $w_2c+c = (w_1d+d)w_3$, or equivalently,

$$\frac{c}{d} = \frac{w_3(w_1+1)}{w_2+1}.$$

If follows that the equations have a unique solution up to scaling (*i.e.* replacing A with λA). Since $\varphi_{\lambda A} = \varphi_A$, there is a unique Möbius transformation with the desired properties. □

Let $\mathcal{C} := \{S \subset \mathbb{C}_\infty \mid S \text{ is a Euclidean circle or a Euclidean line} \cup \{\infty\}\}$.

Lemma. *$S \in \mathcal{C}$ if and only if it is the set of solutions to an equation of the form*

$$\alpha z\overline{z} + (\overline{b}z + \beta\overline{z}) + \gamma = 0$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$, and the equation has more than solution.

Proof. A Euclidean circle with center a and radius r satisfies the equation $|z-a|^2 = r^2$, or equivalently,

$$z\overline{z} - a\overline{z} - \overline{a}z + (|a|^2 - r^2) = 0$$

which is of the desired form. Similarly, if $z = x + iy$, the line $lx + my = n$ ($l, m, n \in \mathbb{R}$) satisfies the equation

$$(l - im)z + (l + im)\overline{z} - 2c = 0.$$

Conversely, given an equation of the form

$$\alpha z\overline{z} + (\overline{b}z + \beta\overline{z}) + \gamma = 0,$$

either $\alpha \neq 0$, in which case we may divide and assume $\alpha = 1$, or $\alpha = 0$. In the first case, we have

$$|z + \beta|^2 = |\beta|^2 - \gamma$$

which is the equation of either a circle, a point, or the empty set. In the second, we have

$$(\operatorname{Re}\beta) + (\operatorname{Im}\beta)y = \gamma/2$$

which is the equation of a line. □

Proposition. *If $C \in \mathcal{C}$ and $\varphi \in \mathrm{Mob}$, then $\varphi(C) \in \mathcal{C}$.*

Proof. Suppose that C satisfies an equation of the form

$$\alpha z\overline{z} + (\overline{b}z + \beta\overline{z}) + \gamma = 0.$$

If $z = \varphi_A(w)$, then w satisfies

$$\alpha' w\overline{w} + (\overline{b}' w + \beta'\overline{w}) + \gamma' = 0,$$

where

$$\begin{aligned}
\alpha' &= \alpha|a|^2 + (\beta\overline{a}c + \overline{\beta}a\overline{c}) + \gamma|c|^2 \\
\beta' &= \alpha b\overline{a} + \beta\overline{a}d + \overline{\beta}a\overline{c} + \gamma d\overline{c} \\
\gamma' &= \alpha|b|^2 + (\beta\overline{b}d + \overline{\beta}b\overline{d}) + \gamma|d|^2.
\end{aligned}$$

Moreover, φ_A is a bijection, so if C contains 0 or 1 points, so does $\varphi_A(C)$. □