Geodesic Polar Coordinates

First, let us recall the definition. If g is a Riemannian metric on U, and $\mathbf{p} \in U$, we choose a g-orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $T_{\mathbf{p}}U$, and let

$$\mathbf{v}_\theta = \cos\theta\, \mathbf{v}_1 + \sin\theta\, \mathbf{v}_2.$$

There is a unique geodesic $\gamma_{\mathbf{v}_\theta}$ with $\gamma_{\mathbf{v}_\theta}(0) = \mathbf{p}$, $\gamma'_{\mathbf{v}_\theta}(0) = \mathbf{v}_\theta$, and we let

$$\varphi_{\mathbf{p}}(r,\theta) = \gamma_{\mathbf{v}_\theta}(r)$$

where this is defined. We view $\varphi_{\mathbf{p}}$ as a map from an open subset of $(0,\infty) \times S^1$ to U. Our first goal is to show there is a smaller open subset $V \subset (0,\infty) \times S^1$ such that $\varphi_{\mathbf{p}}|_V$ is a diffeomorphism onto its image.

To do this, we define a new map $\psi_{\mathbf{p}}$, whose domain is an open subset of $T_{\mathbf{p}}U$. Specifically, we let

$$\psi_{\mathbf{p}}(\mathbf{v}) = \gamma_{\mathbf{v}}(1)$$

where $\gamma_{\mathbf{v}}$ is the unique geodesic with $\gamma_{\mathbf{v}}(0) = \mathbf{p}$, and $\gamma'_{\mathbf{v}}(0) = \mathbf{v}$. (For reasons we won't get into here, $\psi_{\mathbf{p}}$ is usually known as the exponential map.)

Lemma. *$D_{\mathbf{0}}(\psi_{\mathbf{p}})$ is the identity map.*

Proof.

$$\begin{aligned} D_{\mathbf{0}}\psi_{\mathbf{p}}(\mathbf{v}) &= \lim_{\epsilon\to 0} \frac{\psi_{\mathbf{p}}(\epsilon\mathbf{v}) - \psi_{\mathbf{p}}(\mathbf{0})}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\gamma_{\epsilon\mathbf{v}}(1) - \mathbf{p}}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\gamma_{\mathbf{v}}(\epsilon) - \mathbf{p}}{\epsilon} \\ &= \gamma'_{\mathbf{v}}(0) = \mathbf{v} \end{aligned}$$

□

Corollary. *There's an open set $V_1 \subset T_{\mathbf{p}}U$ containing $\mathbf{0}$ such that $\psi_{\mathbf{p}}|_{V_1}$ is a diffeomorphism onto its image.*

Proof. The map $\psi_{\mathbf{p}}$ is a smooth function of $\mathbf{v}$. This follows from a result in the theory of differential equations which we won't try to justify here; namely that if we have a system of ordinary differential equations $\gamma'(t) = F(\gamma(t))$, where $F : \mathbb{R}^n \to \mathbb{R}$ is smooth, then the solutions to this system depend smoothly on the initial conditions. So we can apply the inverse function theorem to $\psi_{\mathbf{p}}$ at $\mathbf{0}$ to get the statement of the corollary. □

Given $\epsilon > 0$, let $V(\epsilon) = (0,\epsilon) \times S^1 \subset (0,\infty) \times S^1$.

Theorem. *We can find $\epsilon > 0$ so that $\varphi_{\mathbf{p}}|_{V(\epsilon)}$ is a diffeomorphism onto its image.*

Proof. Note that $\varphi_{\mathbf{p}}(r,\theta) = \gamma_{\mathbf{v}_\theta}(r) = \gamma_{r\mathbf{v}_\theta}(1) = \psi_{\mathbf{p}}(r\mathbf{v}_\theta)$. So $\varphi_{\mathbf{p}} = \psi_{\mathbf{p}} \circ P$, where $P : (0,\infty)\times S^1 \to (T_{\mathbf{p}}U - \mathbf{0})$ given by $P(r,\theta) = r\mathbf{v}_\theta$ is the ordinary polar coordinates map, which is a diffeomorphism. Choose $\epsilon > 0$ so that the ball $B_\epsilon(\mathbf{0}) = \{\mathbf{v} \in T_{\mathbf{p}}U \,|\, |\mathbf{v}|_{g_{\mathbf{p}}} < \epsilon\}$ is contained in the set V_1 given by the corollary. Then P gives a diffeomorphism from $V(\epsilon)$ to $B_\epsilon(\mathbf{0}) - \mathbf{0}$. Since the composition of diffeomorphisms is a diffeomorphism, $\varphi_{\mathbf{p}}(\mathbf{v})|_{V(\epsilon)}$ is a diffeomorphism onto its image. □

Let $W = \psi_{\mathbf{p}}(B_\epsilon(\mathbf{0}))$. One useful way of rephrasing the theorem is

Corollary. *Any point $\mathbf{q} \in W$ is joined to $\mathbf{p}$ by a unique geodesic of length $< \epsilon$.*

Proof. Geodesics have constant speed, so the length of $\gamma_{\mathbf{v}}|_{[0,1]}$ is $|\mathbf{v}|_{g_{\mathbf{p}}}$. Thus $\mathbf{q}$ is joined to $\mathbf{p}$ by a geodesic of length $< \epsilon$ if and only if $\mathbf{q} = \psi_{\mathbf{p}}(\mathbf{v})$ for $\mathbf{v} \in B_\epsilon(\mathbf{0})$. $\square$

We can use the exponential map to prove a stronger version of this result.

Theorem. *There is an open set $W' \subset U$, $\mathbf{p} \in W'$ and a constant $\epsilon > 0$ such that any two points $\mathbf{q}, \mathbf{r} \in W'$ are joined by a unique geodesic of length $< \epsilon$.*

Proof. We consider a map Ψ which takes an open subset of U to $U \times U$. Ψ is defined by

$$\Psi(\mathbf{q}, \mathbf{v}) = (\mathbf{q}, \psi_{\mathbf{q}}(\mathbf{v})).$$

The derivative $D_{(\mathbf{p},\mathbf{0})}\Psi$ is a 4×4 matrix which can be written in block form as

$$D_{(\mathbf{p},\mathbf{0})}\Psi = \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}$$

The first row in this matrix is easily computed; the I entry in the second row is the derivative of $\psi_{\mathbf{p}}(\mathbf{v})$ with respect to $\mathbf{v}$, which was computed in the lemma.

It follows that $D_{(\mathbf{p},\mathbf{0})}$ is invertible, and we can apply the inverse function theorem to see that there is an open subset $V \subset TU$ containing $(\mathbf{p}, \mathbf{0})$ such that $\Psi|_V$ is a diffeomorphism onto its image.

Since V is open, we can find an open set $U_1 \subset U$ with $\mathbf{p} \in U_1$ and an $\epsilon > 0$ so that the set $V_1 = \{(\mathbf{q}, \mathbf{v}) \in TU \,|\, \mathbf{q} \in U_1, |\mathbf{v}|_{g_{\mathbf{q}}} < \epsilon\}$ is contained in V. Let $W_1 = \Psi(V_1)$. W_1 is open in $U \times U$ and contains $\psi(\mathbf{p}, \mathbf{0}) = (\mathbf{p}, \mathbf{p})$, so we can find an open set $W' \subset U$ containing $\mathbf{p}$ such that $W' \times W' \subset W_1$.

We claim that W' has the desired property. Indeed, if $\mathbf{q}, \mathbf{r} \in W'$, then $(\mathbf{q}, \mathbf{r}) \in W_1$, so $(\mathbf{q}, \mathbf{r}) = \Psi(\mathbf{q}, \mathbf{v})$ for some $(\mathbf{q}, \mathbf{v}) \in V_1$. It follows that $\mathbf{r} = \psi_{\mathbf{q}}(\mathbf{v})$. Since $(\mathbf{q}, \mathbf{v}) \in V_1$, $|\mathbf{v}|_{g_{\mathbf{q}}} < \epsilon$. Thus $\mathbf{r}$ is joined to $\mathbf{q}$ by a geodesic of length $< \epsilon$.

To see that this geodesic is unique, suppose that $\mathbf{r} = \psi_{\mathbf{q}}(\mathbf{v}')$ with $|\mathbf{v}'|_{g_{\mathbf{q}}} < \epsilon$. Then $(\mathbf{q}, \mathbf{v}') \in V_1 \subset V$, and $\Psi(\mathbf{q}, \mathbf{v}) = \Psi(\mathbf{q}, \mathbf{v}') = (\mathbf{q}, \mathbf{r})$. But $\Psi|_V$ is a diffeomorphism onto its image, so this implies that $\mathbf{v} = \mathbf{v}'$.

$\square$