Riemannian metrics define ordinary metrics

Recall that if g is a Riemannian metric on a path-connected open set $U \subset \mathbb{R}^2$, we define the distance between $\mathbf{p}$ and $\mathbf{q}$ by

$$d(\mathbf{p}, \mathbf{q}) = \inf_{\gamma \in \mathcal{P}(\mathbf{p},\mathbf{q})} L(\gamma)$$

where the inf runs over the set of all piecewise smooth paths γ from $\mathbf{p}$ to $\mathbf{q}$. Our aim here is to show that d satisfies the axioms for a metric space.

Proposition 1. *For all $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ in U,*

(1) $d(\mathbf{p}, \mathbf{q}) \geq 0$, *with equality if and only if* $\mathbf{p} = \mathbf{q}$.
(2) $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$.
(3) $d(\mathbf{p}, \mathbf{r}) \leq d(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{r})$

Proof. Somewhat unusually, item (1) is the most difficult to prove. Since $L(\gamma) \geq 0$ for any curve γ, it is clear that $d(\mathbf{p}, \mathbf{q}) \geq 0$. The tricky part is to show that $d(\mathbf{p}, \mathbf{q}) > 0$ if $\mathbf{p} \neq \mathbf{q}$. To do this, we compare g with the standard Euclidean metric on $\mathbb{R}^2$. For $\mathbf{p} \in U$, let $B_\epsilon(\mathbf{p})$ be the closed ball of radius ϵ around $\mathbf{p}$ with respect to the Euclidean metric.

Lemma. *Given $\mathbf{p} \in U$, we can find $\epsilon, \lambda > 0$ so that for any $\mathbf{q}$ in the ball $B_\epsilon(\mathbf{p})$, we have $g_{\mathbf{q}}(\mathbf{u}, \mathbf{u}) \geq \lambda(\mathbf{u} \cdot \mathbf{u})$.*

Proof. The metric g is a smooth map from U to BS_2^+, which is the set of positive-definite, symmetric bilinear forms on $\mathbb{R}^3$. Under the identification

$$(a, b, d) \mapsto \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

we view BS_2^+ as an open subset of $\mathbb{R}^3$ equipped with the Euclidean metric. Thus for any $A \in BS_2^+$, we can find some $\eta > 0$ so that the ball $B_\eta(A)$ is contained in BS_2^+. Now if $\lambda < 2^{-1/2}\eta$, $A - \lambda I \in B_\eta(A)$. Using the triangle inequality, we see that if $A' \in B_{\eta/2}(A)$ and $\lambda < 2^{-3/2}\eta$, then $A' - \lambda I \in BS_2^+$.

Now given $\mathbf{p} \in U$, let $A = g_{\mathbf{p}}$. Choose η as above, and take $\lambda < 2^{-3/2}\eta$. The map g is continuous, so we can find $\epsilon > 0$ so that $g_{\mathbf{q}} \in B_{\eta/2}(A)$ whenever $\mathbf{q} \in B_\epsilon(\mathbf{p})$. It follows that for such $\mathbf{q}$, $g_{\mathbf{q}} - \lambda I$ is positive definite. This means that $(g_{\mathbf{q}} - \lambda I)(\mathbf{u}, \mathbf{u}) \geq 0$ for all $\mathbf{u}$, or equivalently, that $g_{\mathbf{q}}(\mathbf{u}, \mathbf{u}) \geq \lambda(\mathbf{u} \cdot \mathbf{u})$. □

Now let $\gamma : [a, b] \to B_\epsilon(\mathbf{p})$ be a smooth path. If $L_g(\gamma)$ and $L(\gamma)$ are the lengths of γ with respect to g and the Euclidean metric, then

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt \geq \int_a^b \lambda^{1/2} |\gamma'(t)| dt = \lambda^{1/2} L(\gamma).$$

The same inequality clearly holds if γ is piecewise smooth.

Finally, suppose $\gamma : [a, b] \to U$ is a path from $\mathbf{p}$ to $\mathbf{q}$. If the image of γ is contained in $B_\epsilon(\mathbf{p})$, then

$$L_g(\gamma) \geq \lambda^{1/2} L(\gamma) \geq \lambda^{1/2} |\mathbf{q} - \mathbf{p}|.$$

On the other hand, if γ exits $B_\epsilon(\mathbf{p})$, the intermediate value theorem implies there must be some point $t \in [a, b]$ with $d(\gamma(t)) = \epsilon/2$. The set of all such t is closed and contained in $[a, b]$, so it has a minimum value t_0. Then

$$L_g(\gamma) \geq L_g(\gamma|_{[a,t_0]}) \geq \lambda^{1/2} L(\gamma|_{[a,t_0]}) \geq \lambda^{1/2} |\gamma(t_0) - \mathbf{p}| = \lambda^{1/2} \epsilon/2.$$

To sum up, we see that $L_g(\gamma) \geq \lambda^{1/2} \min(|\mathbf{p} - \mathbf{q}|, \epsilon/2)$. Thus for $\mathbf{p} \neq \mathbf{q}$, the inf is strictly positive.

The other two conditions are much easier. For (2), note that if $\gamma : [a,b] \to U$ is a path from $\mathbf{p}$ to $\mathbf{q}$, then the path $\overline{\gamma} : [a,b] \to U$ given by $\overline{\gamma}(s) = \gamma(b+a-s)$ is a path from $\mathbf{q}$ to $\mathbf{p}$ with the same length. So the sets $\{L_g(\gamma) \,|\, \gamma \in \mathcal{P}(\mathbf{p},\mathbf{q})\}$ and $\{L_g(\gamma) \,|\, \gamma \in \mathcal{P}(\mathbf{q},\mathbf{p})\}$ are the same, and thus have the same inf.

Finally, for (3), note that if $\gamma_1 : [a_1, b_1] \to U$ is a piecewise smooth path from $\mathbf{p}$ to $\mathbf{q}$, and $\gamma_2 : [a_2, b_2] \to U$ is a piecewise smooth path from $\mathbf{q}$ to $\mathbf{r}$, then their concatenation $\gamma : [a_1, b_1 + b_2 - a_2] \to U$, defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \leq b_1 \\ \gamma_2(t - b_1 + a_2) & t \geq b_1 \end{cases}$$

is a piecewise smooth path from $\mathbf{p}$ to $\mathbf{r}$, and $L_g(\gamma) = L_g(\gamma_1) + L_g(\gamma_2)$.

Now pick $\gamma_1\,\gamma_2$ as above, with $L_g(\gamma_1) \leq d(\mathbf{p},\mathbf{q}) + \epsilon$ and $L_g(\gamma_2) \leq d(\mathbf{q},\mathbf{r}) + \epsilon$. Then $d(\mathbf{p},\mathbf{r}) \leq L(\gamma) \leq d(\mathbf{p},\mathbf{q}) + d(\mathbf{q},\mathbf{r}) + 2\epsilon$. Since this holds for any $\epsilon > 0$, $d(\mathbf{p},\mathbf{r}) \leq d(\mathbf{p},\mathbf{q}) + d(\mathbf{q},\mathbf{r})$ as desired.

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