**IB GEOMETRY** **LENT 2012**

# EXAMPLE SHEET 3

1. Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$$S(u,v) = \frac{(2u, 2v, u^2+v^2-1)}{1+u^2+v^2}.$$
Show that $S$ defines a parametrized surface whose image is contained in $S^2$.

2. Using the chart from the previous exercise, verify that the tangent space to $S^2$ at a point $\mathbf{x}$ in the image of $S$ is $\mathbf{x}^\perp$.

3. Find an atlas of charts on $S^2$ for which each chart preserves area, and the transition functions relating charts have derivatives with determinant 1. (Hint: consider the circumscribed cylinder.)

4. Using the geodesic equations, show directly that the geodesics in the hyperbolic plane are hyperbolic lines parametrized with constant speed. (Hint: first consider vertical lines in the upper half-pane model.)

5. Let $\Sigma$ be the cylinder $\Sigma = \{(x,y,z) \mid x^2+y^2=1\}$. Prove that $\Sigma$ is locally isometric to the Euclidean plane. Show all geodesics on $\Sigma$ are spirals of the form $\gamma(t) = (\cos at, \sin at, bt)$ where $a^2+b^2=1$.

6. For $a>0$, let $\Sigma$ be the circular half-cone $\Sigma = \{(x,y,z) \mid z^2 = a(x^2+y^2), z>0\}$. Show that $\Sigma$ minus a ray through the origin is locally isometric to the Euclidean plane. When $a=3$, give an explicit formula for the geodesics on $S$ and show that no geodesic intersects itself. For $a<3$ show that there are geodesics which intersect themselves.

7. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $\Sigma \subset \mathbb{R}^3$ be its graph. Show that $\Sigma$ is an embedded surface, and that its Gauss curvature at the point $(x,y,F(x,y))$ is the value of
$$\frac{F_{xx}F_{yy} - F_{xy}^2}{(1+F_x^2+F_y^2)^2}$$
at the point $(x,y)$.

8. Let $\gamma$ be an embedded curve in the $xz$-plane given by the parametrization $\gamma(t) = (f(t), 0, g(t))$, where $f(t)>0$ for all $t$, and let $\Sigma$ be the surface obtained by rotating $\gamma$ around the $z$-axis. Show that the Gauss curvature of $\Sigma$ is
$$K = \frac{(\dot f\ddot g - \ddot f\dot g)\dot g}{f(\dot f^2+\dot g^2)^2}.$$
If $\gamma$ is parametrized so as to have unit speed ($\dot f^2 + \dot g^2 = 1$), show that this reduces to $K = -\ddot f/f$.

9. Using the previous question, compute the Gauss curvature of the surfaces given by the equations $x^2+y^2-z^2=1$ and $x^2+y^2-z^2=-1$. Describe the qualitative properties of the curvature in these cases (sign and behavior near $\infty$) and explain what you find using pictures of these surfaces.

10. Let $T$ be the torus obtained by rotating the circle in the $xz$-plane given by the equation $(x-2)^2 + z^2 = 1$ around the $z$-axis. Find the Gauss curvature $K$ of $T$, and identify the points on $T$ where $K$ is positive, negative, and zero. Verify that

$$\int_T K \, dA = 0.$$

11. Let $D$ be an open disc centered at the origin in $\mathbb{R}^2$. Give $D$ a Riemannian metric of the form $(dx^2 + dy^2)/f(r)^2$, where $r = \sqrt{x^2 + y^2}$ and $f(r) > 0$. Show that the curvature of this metric is $K = ff'' - (f')^2 + ff'/r$.

12. Show that the embedded surface given by the equation $x^2 + y^2 + c^2 z^2 = 1$ $(c > 0)$ is homeomorphic to $S^2$. Deduce from the global Gauss-Bonnet theorem that

$$\int_0^1 (1 + (c^2 - 1)u^2)^{-3/2} du = c^{-1}.$$

Can you verify this formula directly?

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13. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve in the plane with $\|\gamma'(t)\| = 1$, and let $\mathbf{n}$ be the unit normal vector obtained by rotating $\gamma'(t)$ counterclockwise by an angle of $\pi/2$. Show that $\gamma''(t) = \kappa(t)\mathbf{n}$ for some function $\kappa(t)$. $\kappa(t)$ is called the curvature of $\gamma$ at the point $\gamma(t)$. If $C(t)$ is the circle which is tangent to second-order to $\gamma$ at $\gamma(t)$, show that the radius of $C(t)$ is $1/|\kappa(t)|$. If the image of $\gamma$ is a graph $(x, f(x))$ with $f(0) = f'(0) = 0$, show that the curvature at $(0, 0)$ is $f''(0)$.

14. Suppose $\Sigma$ is a surface of revolution obtained by rotating a curve $\gamma$ in the $xz$-plane about the $z$-axis. Find $\gamma$ such that the Gauss curvature of $\Sigma$ is identically $-1$.

15. Let $\Sigma$ be a compact embedded surface in $\mathbb{R}^3$. By considering the smallest closed ball centered at the origin which contains $\Sigma$, show that the Gauss curvature must be strictly positive at some point of $\Sigma$. Conclude that the locally Euclidean metric on the torus cannot obtained as the first fundamental form of a smoothly embedded torus in $\mathbb{R}^3$.

16. Show that a genus two surface can be obtained by appropriately identifying the sides of a regular octogon. Using problem 10 on example sheet 2, show that the genus two surface admits a Riemannian metric with constant curvature $K = -1$. Explain how to generalize your argument to arbitrary surfaces of genus $g > 1$.

**Note to the reader:** You should look at all questions up to question (12), and then any further questions you have time for.

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