IB GEOMETRY LENT 2012

EXAMPLE SHEET 2

1. For each map $S : U \to \mathbb{R}^3$, find the Riemannian metric on U induced by S. Sketch the image of S in $\mathbb{R}^3$.

   (a) $U = (0, 2\pi) \times \mathbb{R}, S(\theta, z) = (\cos\theta, \sin\theta, z)$.

   (b) $U = (0, 2\pi) \times (-1, 1), S(\theta, z) = (\cos\theta\sqrt{1 - z^2}, \sin\theta\sqrt{1 - z^2}, z)$.

   (c) $U = (0, 2\pi) \times (0, 2\pi), S(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ where $a > b$.

2. Define a Riemannian metric on the unit disk $D \subset \mathbb{C}$ by $(du^2 + dv^2)/(1 - u^2 - v^2)$. Prove that the diameters are length-minimizing curves for this metric. Show that distances in this metric are bounded, but areas can be unbounded.

3. Let $V \subset \mathbb{R}^2$ be the square $|u|, |v| < 1$, and define two Riemannian metrics on V by

$$\frac{du^2}{(1 - u^2)^2} + \frac{dv^2}{(1 - v^2)^2} \quad \text{and} \quad \frac{du^2}{(1 - v^2)^2} + \frac{dv^2}{(1 - u^2)^2}.$$

   Prove that there is no isometry between the spaces, but that there is an area preserving diffeomorphism between them. (*Hint:* show that in one space there are curves of finite length going out to the boundary, while in the other no such curves exist.)

4. Let H denote the upper half-plane model of hyperbolic space. If l is the hyperbolic line in H given by a Euclidean semicircle with center $a \in \mathbb{R}$ and radius $r > 0$, show that reflection in the line l is given by the formula

$$R_l(z) = a + \frac{r^2}{\overline{z} - a}.$$

5. If a is a point in the upper half-plane, show that the Mobius transformation ϕ given by $\phi(z) = (z - a)/(z - \overline{a})$ defines an isometry from H to the disk model D of the hyperbolic plane. Deduce that for points $z_1, z_2 \in H$, the hyperbolic distance is given by $\rho(z_1, z_2) = 2\tanh^{-1}|(z_1 - z_2)/(z_1 - \overline{z}_2)|$.

6. Let z_1, z_2 be distinct points in H. Suppose that the hyperbolic line through z_1 and z_2 meets the real axis at points w_1 and w_2, where z_1 lies on the hyperbolic line segment $w_1 z_2$ and one of w_1 or w_2 might be ∞. Show that the hyperbolic distance $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, w_1, w_2, z_2 taken in an appropriate order.

7. Let C be a hyperbolic circle in H; show that C is also a Euclidean circle. If C has hyperbolic center ic ($c \in \mathbb{R}^+$) and hyperbolic radius ρ, find the radius and center of C regarded as a Euclidean circle. Find the hyperbolic area and perimeter of C.

8. Prove that two different hyperbolic circles intersect in at most two points. If $\triangle ABC$ and $\triangle A'B'C'$ are hyperbolic triangles whose sides have equal hyberbolic lengths, show there is an isometry of H which takes $\triangle ABC$ to $\triangle A'B'C'$.

9. Given two points P and Q in the hyperbolic plane, show that the locus of points equidistant from P and Q is a hyperbolic line.

10. Prove that a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$ has area $(n-2)\pi - \sum \alpha_i$. Show that for every $n \geq 3$ and every α with $0 \leq \alpha \leq (1 - \frac{2}{n})\pi$, there is a regular n-gon all of whose angles are α.

11. Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the perpendicular is unique. Given two ultraparallel hyperbolic lines, prove that the composition of the corresponding reflections has infinite order. (*Hint:* you may wish to take the common perpendicular as a special line.)

12. Let M be the hyperboloid model of the hyperbolic plane. That is, consider the Lorentzian inner product $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$ on $\mathbb{R}^3$, and let $M = \{\mathbf{x} \in \mathbb{R}^3 \mid \langle\langle \mathbf{x}, \mathbf{x} \rangle\rangle = -1, x_3 > 0\}$ with the Riemannian metric restricted from $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle$. Show that any plane P in $\mathbb{R}^3$ through the origin that meets M can be written as $\{\mathbf{x} \in \mathbb{R}^3 \mid \langle\langle \mathbf{x}, \mathbf{u} \rangle\rangle = 0\}$ for some $\mathbf{u} \in \mathbb{R}^3$ with $\langle\langle \mathbf{u}, \mathbf{u} \rangle\rangle = 1$. Use this to write a formula for the reflection of M in the hyperbolic line $M \cap P$. Show that every hyperbolic line of M arises in this way.

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13. Fix a point P on the boundary of D, the unit disk model of the hyperbolic plane. Determine which curves in D are orthogonal to every hyperbolic line that passes through P.

14. Show that there is a constant k such that no hyperbolic triangle contains a circle of radius greater than k. Conclude that there is another constant k' so that if $\triangle ABC$ is any hyperbolic triangle, then any point on $\overline{BC}$ is within hyperbolic distance k' of either $\overline{AB}$ or $\overline{AC}$.

15. Suppose ϕ is an orientation preserving isometry of the hyperbolic plane, which we will view in the unit disk model. Show that either a) ϕ fixes a point in the interior of D, b) ϕ fixes two points on ∂D or c) ϕ fixes one point P on ∂D. Show that in case a) ϕ is a rotation, in b) that it fixes a hyperbolic line, and in c) that it fixes any Euclidean circle tangent to ∂D at P.

16. For $z, w \in \mathbb{C}$, show that $|1 - z\overline{w}|^2 = |z - w|^2 + (1 - |z|^2)(1 - |w|^2)$. Given points z, w in D, prove that
$$\sinh^2\left(\frac{1}{2}\rho(z, w)\right) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}$$
where ρ denotes the hyperbolic distance.

Note to the reader: You should look at all questions up to question (12), and then any further questions you have time for.

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