IB GEOMETRY **LENT 2012**

EXAMPLE SHEET 1

1. Suppose that l_1 and l_2 are non-parallel lines in $\mathbb{R}^2$, and that $R_i : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the reflection in the line l_i for $i = 1, 2$. Show that the composition $R_1 R_2$ is a rotation. Describe the center and angle of rotation in terms of l_1 and l_2.

2. Suppose that $\phi \in \mathrm{Isom}(\mathbb{R}^2)$. Show that there is either a point $x \in \mathbb{R}^2$ with $\phi(x) = x$ or a line $l \subset \mathbb{R}^2$ with $\phi(l) = l$. Conclude that ϕ is either (a) a translation, (b) a rotation, (c) a reflection, or (d) a composition $R \circ T$, where R is reflection in a line l and T is translation by some nonzero vector.

3. Suppose that H is a hyperplane in $\mathbb{R}^n$ defined by the equation $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant c. The reflection in H is the map from $\mathbb{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u} - c)\mathbf{u}$. Show this is an isometry. If P and Q are points of $\mathbb{R}^n$, show that reflection in some hyperplane maps P to Q.

4. Suppose that P_1, P_2 are points in $\mathbb{R}^2$, and that $a_1, a_2 \in \mathbb{R}$. Show that there are at most two points $Q \in \mathbb{R}^2$ with $d(P_i, Q) = a_i$. If Δ_1, Δ_2 are two triangles in $\mathbb{R}^2$ with the same side lengths, show there is a $\phi \in \mathrm{Isom}(\mathbb{R}^2)$ with $\phi(\Delta_1) = \Delta_2$.

5. Let G be a finite subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. By considering the barycentre (*i.e.* average) of the orbit of the origin under G, show that G fixes some point of $\mathbb{R}^n$. If $n = 2$, show that G is either cyclic or *dihedral* (that is $D_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and for $n \geq 3$, D_{2n} is the full symmetry group of a regular $2n$-gon.)

6. Prove that any matrix $A \in O(3, \mathbb{R})$ is the product of at most three reflections in planes through the origin. Deduce that an isometry of the unit sphere can be obtained as the product of at most three reflections in spherical lines. Which isometries are obtained as the product of two reflections? Which are the product of three reflections and no fewer?

7. Let Δ be a spherical triangle with sides of length a, b, c and opposite angles α, β, γ. Extend the sides of Δ to form complete great circles. Show that this divides the sphere into 8 triangles and find the side lengths and angles for each.

8. In the spherical triangle $\Delta = ABC$ show that $b = c$ if and only if $\beta = \gamma$. Show that this occurs if and only if there is a reflection which exchanges the sides of length b and c. Are there equilateral spherical triangles? Are they all isometric to one another?

9. Let P be a point on the unit sphere S^2. For fixed $0 < \rho < \pi$, the *spherical circle* with centre P and radius ρ is the set of points $Q \in S^2$ whose spherical distance from P is ρ. Prove that a spherical circle of radius ρ has circumference $2\pi \sin \rho$ and area $2\pi(1 - \cos \rho)$. Deduce that the map from the cylinder of radius one to S^2 given (in cylindrical coordinates) by $(1, \theta, z) \mapsto (\sqrt{1 - z^2}, \theta, z)$ is area preserving.

10. Prove that Mobius transformations of $\mathbb{C}_\infty$ preserve cross ratios. If $u, v, \in \mathbb{C}$ correspond to points P, Q on S^2, and d denotes the angular distance from P to Q on S^2, show that $-\tan^2(d/2)$ is the cross ratio of the points $u, v, -1/\overline{u}, -1/\overline{v}$, taken in an appropriate order.

11. Show that any Möbius transformation $T \neq 1$ on $\mathbb{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of S^2 through a nonzero angle has exactly two fixed points z_1 and $z_2 = -1/\overline{z}_1$. If T is a Möbius transformation with two fixed points z_1 and $z_2 = -1/\overline{z}_1$, show that either T corresponds to a rotation of S^2, or one of the fixed points — say z_1 — is an attracting fixed point; that is for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

12. Suppose we have a polygonal decomposition of S^2 by convex geodesic polygons, where each polygon is contained in some hemisphere. Denote by F_n the number of faces with precisely n edges, and V_m the number of vertices where precisely m edges meet; show that $\sum_n nF_n = 2E = \sum_m mV_m$.

    Suppose that $V_i = F_i = 0$ for $i < 3$. If in addition $V_3 = 0$, deduce that $E \geq 2V$. Similarly, if $F_3 = 0$, deduce that $E \geq 2F$. Conclude that $V_3 + F_3 > 0$. Prove the identity
$$\sum_n (6-n)F_n = 12 + 2\sum_m (m-3)V_m.$$
    Deduce that $3F_3 + 2F_4 + F_5 \geq 12$. The surface of a football is decomposed into spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there?

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13. A spherical triangle $\Delta = ABC$ has vertices given by unit vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbb{R}^3$, sides of length a, b, c, and angles α, β, γ. The *polar triangle* $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$, and $\mathbf{B} \times \mathbf{A}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$, and $\pi - a$, $\pi - b$ and $\pi - c$, respectively. Deduce that
$$\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha \cos\beta.$$

14. Exhibit a subset X of $\mathbb{R}^2$ such that (a) any two points $x, y \in X$ can be joined by a continuous path $\gamma : [0,1] \to X$ and (b) for $x \neq y$ the length of any such path is infinite.

15. Let v be a vertex of a convex Euclidean polyhedron P. For each face f containing v, let $\theta_v(f)$ be the angle of f with vertex v. Prove that $\sum_f \theta_v(f) < 2\pi$. If we define $d(v) = 2\pi - \sum_f \theta_v(f)$, show that $\sum_v d(v) = 4\pi$, where the sum runs over all vertices of P.

    A regular polyhedron is one in which all two-dimensional faces are congruent regular polygons, and such that for each pair of vertices v and v' there is some $\phi \in \mathrm{Isom}(\mathbb{R}^3)$ with $\phi(P) = P$ and $\phi(v) = v'$. Show that there are five types of regular polyhedron, and compute the number of vertices, edges, and faces for each type.

16. The Euler characteristic of an n-dimensional convex polyhedron P is $\chi_n(P) = \sum_i (-1)^i F_i$, where F_i denotes the number of i-dimensional faces. Compute χ for the n-dimensional analogs of the tetrahedron and the cube. Assuming that the value of $\chi_n(P)$ does not depend on the choice of convex polyhedron P, prove that $\chi_n(P) = 0$ for n odd.

Note to the reader: You should look at all questions up to question (12), and then any further questions you have time for.

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