GEODESIC POLAR COORDINATES

Suppose $U \subset \mathbb{R}^2$ is an open set, and g is a Riemannian metric on U. Let p be a point of U, and let $\mathbf{v} \in T_pU = \mathbb{R}^2$ be a tangent vector. We saw previously that there is a unique geodesic $\gamma_{\mathbf{v}}$ with $\gamma_{\mathbf{v}}(0) = p$ and $\gamma_{\mathbf{v}}'(0) = \mathbf{v}$.

Lemma. $\gamma_{k\mathbf{v}}(t) = \gamma_{\mathbf{v}}(kt)$.

Proof. Let $\Gamma(t) = \gamma_{\mathbf{v}}(kt)$. Inspecting the equations for a geodesic, we see that they are also satisfied by Γ. Now $\Gamma(0) = \gamma(0) = p$, and $\Gamma'(0) = k\gamma_{\mathbf{v}}'(0) = k\mathbf{v}$, so $\gamma_{k\mathbf{v}}(t) = \Gamma(t)$. □

We now define a map $S : B(\epsilon) \to U$ by $S(\mathbf{v}) = \gamma_{\mathbf{v}}(1)$.

Proposition. *For small enough ϵ, S is a diffeomorphism onto its image.*

Proof. By the Inverse Function theorem, it suffices to show that $DS|_0$ is invertible. We compute

$$\begin{aligned}
DS(\mathbf{w}) &= \lim_{\epsilon \to 0}(S(\epsilon\mathbf{w}) - S(0))/\epsilon \\
&= \frac{d}{d\epsilon}(S(\epsilon\mathbf{w}))|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}(\gamma_{\epsilon\mathbf{w}}(1)) \\
&= \frac{d}{d\epsilon}(\gamma_{\mathbf{w}}(\epsilon)))|_{\epsilon=0} \\
&= \gamma_{\mathbf{w}}'(0) \\
&= \mathbf{w}
\end{aligned}$$

In other words, $DS|_0$ is the identity map, which is certainly invertible. □

Choose a basis $\mathbf{e}_1, \mathbf{e}_2$ for T_pU which is orthonormal with respect to the metric g. Let $p : (0, \epsilon) \times [0, 2\pi) \to B(\epsilon)$ be the map : $p(r, \theta) = r\cos\theta\ \mathbf{e}_1 + r\sin\theta\ \mathbf{e}_2$, and let $T : (0, \epsilon) \times [0, 2\pi) \to U$ be the composition $T = S \circ p$. In other words

$$T(r, \theta) = \gamma_{r\mathbf{v}_\theta}(1) = \gamma_{\mathbf{v}_\theta}(r)$$

where $\mathbf{v}_\theta = \cos\theta\ \mathbf{e}_1 + \sin\theta\ \mathbf{e}_2$. The coordinates on the image of T defined by r and θ are known as *geodesic polar coordinates*. The metric with respect to geodesic polar coordinates takes an especially simple form:

Theorem. $T^*(g) = dr^2 + G(r, \theta)d\theta^2$

Proof. $T^*(g) = Edr^2 + 2Fdrd\theta + Gd\theta^2$, where

$$E = \langle T_r, T_r\rangle_g \quad F = \langle T_r, T_\theta\rangle_g \quad G = \langle T_\theta, T_\theta\rangle_g.$$

We compute

$$\begin{aligned} T_r(r,\theta) &= \frac{\partial}{\partial r} T(r,\theta) \\ &= \frac{d}{dr}\gamma_{\mathbf{v}_\theta}(r) \\ &= \gamma'_{\mathbf{v}_\theta}(r) \end{aligned}$$

Now $\gamma_{\mathbf{v}_\theta}$ is a geodesic, so it has constant speed with respect to the metric g. Thus

$$E = \langle T_r, T_r\rangle_g = \langle \gamma'_{\mathbf{v}_\theta}(r), \gamma'_{\mathbf{v}_\theta}(r)\rangle_g = \langle \gamma'_{\mathbf{v}_\theta}(0), \gamma'_{\mathbf{v}_\theta}(0)\rangle_g = \langle \mathbf{v}_\theta, \mathbf{v}_\theta\rangle_g = 1.$$

To show that $\langle T_r, T_\theta\rangle_g = 0$, observe that γ_θ is a geodesic with unit speed, so the energy

$$\mathcal{E}_{[0,r]}(\gamma_\theta) = \int_0^r \langle \gamma'_{\mathbf{v}_\theta}(t), \gamma'_{\mathbf{v}_\theta}(t)\rangle_g \; dt = \int_0^r dt = r.$$

Thus

$$0 = \frac{\partial}{\partial \theta}(\mathcal{E}_{[0,r]}(\gamma_{\mathbf{v}_\theta})) = D_{\mathbf{V}}\mathcal{E}|_{\gamma_{\mathbf{v}_\theta}}$$

where $\mathbf{V}(t) = \frac{\partial}{\partial\theta}(\gamma_{\mathbf{v}_\theta}(t))$. In particular, $\mathbf{V}(r) = \frac{\partial}{\partial\theta}(\gamma_{\mathbf{v}_\theta}(r)) = T_\theta$.

Now we use the formula for the variation of energy we derived in class:

$$\begin{aligned} D_{\mathbf{V}}\mathcal{E}|_\gamma &= 2[(E\dot\gamma_1 + F\dot\gamma_2)\mathbf{V}_1 + (F\dot\gamma_1 + G\dot\gamma_2)\mathbf{V}_2]_0^r \\ &\quad - 2\int_0^r \frac{d}{dt}(E\dot\gamma_1 + F\dot\gamma_2)\mathbf{V}_1 + \frac{d}{dt}(F\dot\gamma_1 + G\dot\gamma_2)\mathbf{V}_2 \; dt \\ &\quad + \int_0^r (E_u\dot\gamma_1^2 + 2F_u\dot\gamma_1\dot\gamma_2 + G_u\dot\gamma_2^2)\mathbf{V}_1 + (E_v\dot\gamma_1^2 + 2F_v\dot\gamma_1\dot\gamma_2 + G_v\dot\gamma_2^2)\mathbf{V}_2 \; dt \end{aligned}$$

In our case $\gamma = \gamma_{\mathbf{v}_\theta}$ is a geodesic, so the combined contribution from the second and third lines is 0. Moreover, $\gamma_{\mathbf{v}_\theta}(0) \equiv p$, so $\mathbf{V}(0) = 0$. Thus the above expression reduces to

$$\begin{aligned} 0 = D_{\mathbf{V}}\mathcal{E}|_{\gamma_{\mathbf{v}_\theta}} &= 2[(E\dot\gamma_1(r) + F\dot\gamma_2(r))\mathbf{V}_1(r) + (F\dot\gamma_1(r) + G\dot\gamma_2(r))\mathbf{V}_2(r)] \\ &= 2\langle \gamma'_{\mathbf{v}_\theta}(r), \mathbf{V}(r)\rangle_g \\ &= 2\langle T_r(r,\theta), T_\theta(r,\theta)\rangle_g. \end{aligned}$$

This is what we wanted to prove. □

The following result will be helpful in the proof of the Gauss-Bonnet theorem:

Lemma. *$G(r,\theta) = r^2 f(r,\theta)$, where $\lim_{r\to 0} f(r,\theta) = 1$.*

Proof. Let g_1 be any Riemannian metric on $B(\epsilon)$, and let $\mathbf{e}_1, \mathbf{e}_2$ be an orthonormal basis of $T_0(B(\epsilon))$. Define $p : (0,\epsilon)\times(0,2\pi) \to B(\epsilon)$ by $p(r,\theta) = r\cos\theta\,\mathbf{e}_1 + r\sin\theta\,\mathbf{e}_2$, and let $p^*(g_1) = E\,dr^2 + 2F\,drd\theta + G\,d\theta^2$.

We claim that G is of the form stated in the lemma. Indeed,

$$\begin{aligned} G &= \langle p_\theta, p_\theta\rangle_{g_1} \\ &= \langle -r\sin\theta\,\mathbf{e}_1 + r\cos\theta\,\mathbf{e}_2, -r\sin\theta\,\mathbf{e}_1 + r\cos\theta\,\mathbf{e}_2\rangle_{g_1} \\ &= r^2\sin^2\theta\,\langle \mathbf{e}_1,\mathbf{e}_1\rangle_{g_1} - 2r\sin\theta\cos\theta\,\langle \mathbf{e}_1,\mathbf{e}_2\rangle_{g_1} + r^2\cos^2\theta\,\langle \mathbf{e}_2,\mathbf{e}_2\rangle_{g_1}. \end{aligned}$$

Now $\mathbf{e}_1$ and $\mathbf{e}_2$ were an orthonormal basis of $T_0B(\epsilon)$, so as $r \to 0$,

$$\langle \mathbf{e}_1,\mathbf{e}_1\rangle_{g_1} \to 1 \qquad \langle \mathbf{e}_1,\mathbf{e}_2\rangle_{g_1} \to 0 \qquad \langle \mathbf{e}_2,\mathbf{e}_2\rangle_{g_1} \to 1,$$

and the claim follows.

The statement of the lemma follows immediately, since $T^*(g) = p^*(S^*(g))$. $\square$

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