ALGEBRAIC TOPOLOGY (PART III) **MICHELMAS 2009**

EXAMPLE SHEET 4

PART A

1. Suppose $\pi : E \to B$ is a locally trivial fibration, and that $f : X \to B$. Describe transition functions for $f^*(E)$ in terms of transition functions for E.

2. If $\gamma : S^{n-1} \to SO(k)$, let V_γ be the oriented k-dimensional vector bundle over S^n with transition function given by γ. Show that if $\gamma_1 \sim \gamma_2$, then $V_{\gamma_1} \simeq V_{\gamma_2}$. (You may assume that $f_1 \sim f_2$ implies $f_1^*(V) \simeq f_2^*(V)$.)

3. If k is even, show there is an orientation-reversing isomorphism between V_γ and $V_{-\gamma}$.

4. Let $\pi : E_n \to S^2$ be the bundle with fibre S^1 whose transition function is given by a map $S^1 \to SO(2)$ of degree n. For this bundle, describe/compute

   (a) the Leray-Serre spectral sequence on homology and cohomology.

   (b) the Gysin sequence.

   (c) the Thom class and the Euler class.

   Show that E_1 is S^3 and that E_2 is $\mathbb{RP}^3$. More generally, show that if m divides n there is a covering map $E_m \to E_n$. Conclude that the universal cover of E_n is S^3.

5. Use the Euler class to define a homomorphism $\pi_{n-1}(SO(n)) \to \mathbb{Z}$. Conclude that $\pi_{n-1}(SO(n))$ is infinite for even n.

6. If V_1 and V_2 are two vector bundles over B, then $V_1 \times V_2$ is a bundle over $B \times B$. The *Whitney sum* $V_1 \oplus V_2$ is defined to be $\Delta^*(V_1 \times V_2)$, where $\Delta : B \to B \times B$ is the diagonal map. Show that $e(V_1 \oplus V_2) = e(V_1) \cup e(V_2)$.

7. Suppose M is a 4-manifold, and that $S \subset M$ is an embedded sphere with $[S] \cdot [S] = 1$. Show that $M = M' \# \mathbb{CP}^2$ for some M'.

8. Show that up to isomorphism, there are precisely two 3-dimensional real vector bundles over S^2. Use the Leray-Serre spectral sequence to compute the cohomology groups of their associated sphere bundles. Show directly that these sphere bundles are homeomorphic to $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Conclude that the ring structure on cohomology cannot be deduced from the spectral sequence.

PART B

1. Show that real 1-dimensional vector bundles over B are in 1-1 correspondence with elements of $H^1(B;\mathbb{Z}/2)$.

2. (For those who know something about Lie groups.) Suppose G is a Lie group and H is a connected subgroup of G. Prove there is a fibration $G \to G/H$ with fibre H.

3. Show that $SO(3)$ is homeomorphic to $\mathbb{RP}^3$, and that $SO(4)$ is homeomorphic to $S^3 \times S^3/\pm 1$, where the ± 1 acts diagonally on the two factors. (What does this have to do with the Dynkin diagram?) Relate these homeomorphisms to the fibrations $SO(n) \to S^n$ provided by the previous problem.

4. Show that $SU(2)$ is homeomorphic to S^3. Prove inductively that $H^*(SU(n))$ is an exterior algebra on generators of dimensions $3, 5, 7, \ldots 2n-1$. (Is $SU(n)$ homotopy equivalent to a product of spheres?)

5. Classify 3-dimensional real vector bundles over S^4. Show that if V is such a bundle which splits as a Whitney sum ($V = V_1 \oplus V_2$), then V is trivial. Conclude that there exist bundles with zero Euler class which do not admit any nonvanishing section.

6. Show that $\pi_3(SO(4)) \simeq \mathbb{Z}^2$. Conclude that there are infinitely many different bundles $M \to S^4$ with fibre S^3 and $H_*(M) \simeq H_*(S^7)$. (The total spaces of these bundles are all homeomorphic to S^7, but not all of them are diffeomorphic to it.)

7. Show that the Euler class of an odd dimensional real vector bundle is $V \to B$ is a 2-torsion element of $H^*(B)$.

8. The complex *Grassmann variety* $G(k,n)$ is defined to be the set of k-dimensional subspaces of $\mathbb{C}^n$. Show that $G(k,n) \simeq U(n)/(U(k) \times U(n-k))$. Compute its Poincare polynomial.

9. The variety of complete flags in $\mathbb{C}^n$ (written $Fl_n(\mathbb{C})$) is defined to be the set of sequences

$$0 = V_0 \subset V_1 \subset \ldots V_{n-1} \subset V_n = \mathbb{C}^n$$

where each V_i is a complex linear subspace of dimension i. Show that there is fibration $Fl_n(\mathbb{C}) \to \mathbb{CP}^{n-1}$ with fibre $Fl_{n-1}(\mathbb{C})$. Compute the Poincare polynomial of $Fl_n(\mathbb{C})$.

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