EXAMPLE SHEET 3

PART A

1. If X is a finite cell complex with $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}/2, H_2(X) = \mathbb{Z}/4$, and $H_*(X) = 0$ for $* > 2$, compute $H_*(X; G)$ for $G = \mathbb{Z}/2, \mathbb{Z}/4$ and $H^*(X; G)$ for $G = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/4$.
2. For X as above, compute $H_*(X \times X)$.
3. Suppose that M is a connected n-manifold and that $\Sigma^k M$ (the k-fold suspension) is an $n+k$-manifold for some $k > 0$. Show that $H_*(M) \simeq H_*(S^n)$.
4. If M_1 and M_2 are closed connected oriented n-manifolds, $M_1 \# M_2$ is the manifold obtained by removing small n-balls from M_1 and M_2 and identifying their boundaries by a homeomorphism which is orientation reversing with respect to the induced orientations on the boundary spheres.
   (a) Show that there is an orientation on $M_1 \# M_2$ which is compatible with the orientations on M_1 and M_2, in the sense that there are natural maps $p_i : M_1 \# M_2 \to M_i$ with $p_{i*}([M_1 \# M_2]) = [M_i]$.
   (b) Show that $H_*(M_1 \# M_2) \simeq H_*(M_1) \oplus H_*(M_2)$ for $* \neq 0, n$.
   (c) Describe the ring structure on $H^*(M_1 \# M_2)$ in terms of the ring structure on $H^*(M_i)$.
5. Let S_g be the orientable surface of genus g. Show that if $g < h$, then every map $S_g \to S_h$ has degree 0.
6. If M is a simply connected 4-manifold, show that $\chi(M) \geq 2$.
7. Let $\mathbb{CP}^2$ have the orientation coming from its usual structure as a complex manifold, and let $\overline{\mathbb{CP}}^2$ be the same manifold with the opposite orientation. Show that no two of $S^2 \times S^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$, and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ are homotopy equivalent.
8. Show that any map $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ has a fixed point. Construct a map $f : \mathbb{CP}^{2n+1} \to \mathbb{CP}^{2n+1}$ which has no fixed points.

PART B

1. Suppose X and Y are finite cell complexes. A map $f : X \to Y$ is called *cellular* if $f(X_{(n)}) \subset Y_{(n)}$ for all n.
   (a) Show that any $f : X \to Y$ is homotopic to a cellular map. (Hint: Induct on the dimension of X).
   (b) If f is cellular, there is a well-defined map $f_* : H_*(X_{(n)}, X_{(n-1)}) \to H_*(Y_{(n)}, Y_{(n-1)})$. Show that f_* defines a chain map $C_*^{cell}(X) \to C_*^{cell}(Y)$ and that the induced map on cellular homology agrees with $f_* : H_*(X) \to H_*(Y)$.
   (c) Let $\pi : X \times Y \to X$ be the projection and $j : X \to X \times Y$ be the map given by $j(x) = (x, p)$ for a fixed point $p \in Y$. Use the natural cell structure on $X \times Y$ to compute $\pi_*([x] \otimes [y])$ and $\pi^*([a])$, as well as $j_*([x])$ and $j^*([a] \otimes [b])$.
   (d) Using the results of the previous part, show that $1 \cup a = a$ for all $a \in H^*(X)$.

2. Define $f : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ by $f([z_0 : z_1], [w_0 : w_1]) = [z_1 w_0 + z_0 w_1 : z_0 w_0; z_1 w_1]$. Show that f is a 2-1 covering map away from the diagonal. Determine the maps f_* and f^*. Let x be a generator of $H_2(\mathbb{CP}^2)$. Use your answer to the previous question to compute $x \cup x$.

3. If M is an orientable 3-manifold, show that M can be decomposed as $H_g \cup_\phi H_g$, where H_g denotes a handlebody of genus g, (*i.e.* the region inside the standard embedding of S_g in $\mathbb{R}^3$) and $\phi : S_g \to S_g$ is an orientation reversing homeomorphism. (You may assume M has a handle decomposition.)

4. Use the Borsuk-Ulam theorem to prove the "Ham Sandwich Theorem": if $A_1, \ldots, A_n$ are bounded measurable sets in $\mathbb{R}^n$, there is a single hyperplane in R^n which divides each A_i into two equal volumes.

5. Show that the Hopf invariant is additive: if $f_1, f_2 \in \pi_{4n-1}(S^{2n})$, then $H(f_1 + f_2) = H(f_1) + H(f_2)$. Conclude that the groups $\pi_3(S^2)$ and $\pi_7(S^4)$ are infinite.

6. Show that an orientable $(4n+2)$-manifold has even Euler characteristic. (Hint: consider the intersection pairing on $H_{2n+1}(M)$.)

7. Let M be an orientable $4n$-manifold. Show that the intersection pairing defines a nondegenerate quadratic form on $H_{2n}(M; \mathbb{R})$. Recall that nondegenerate quadratic forms on real vector spaces are classified by their *signature*: if the form is represented by a symmetric matrix M, its signature is the number of positive eigenvalues of M minus the number of negative eigenvalues. Show that if M bounds an orientable $(4n+1)$-manifold, then the signature of the associated quadratic form is 0. Conclude that $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not bound an orientable 5-manifold.

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